A CRITERION OF CONVERGENCE OF A BRANCHED CONTINUED FRACTION WITH POSITIVE ELEMENTS

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An inequality for harmonic means is proved and used to establish a sufficient criterion of convergence and to estimate the rate of convergence of branched continued fractions with positive components.

The problem of convergence for continued fractions with positive elements is completely solved by the Seidel criterion [2, 6–8].

Seidel Criterion. *A continued fraction*

$$b_0 + \overset{\infty}{\underset{k=1}{\mathrm{D}}} \frac{1}{b_k},$$

where $b_k > 0$, $k = 1, 2, \ldots$, *converges iff the series* $\sum_{k=1}^{\infty} b_k$ *is divergent.*

Among the great variety of multidimensional generalizations of continued fractions, a significant role is played by branched continued fractions (BCF) introduced by Skorobogat'ko [4]. A branched continued fraction is an analog of continued fractions for functions of many variables.

The necessary, sufficient, and necessary and sufficient criteria for the convergence of the BCF

$$b_0 + \overset{\infty}{\underset{k=1}{\mathrm{D}}} \sum_{i_k=1}^{N} \frac{1}{b_{i(k)}}, \tag{1}$$

where $b_{i(k)} > 0$, $i_k = 1, \ldots, N$, $k = 1, 2, \ldots$, were established in [1, 5].

In particular, it was discovered that fraction (1) diverges if the series $\sum_{k=1}^{\infty} \beta_k$ converges, where $\beta_k = \max(b_{i(k)}, i_p = 1, \ldots, N, p = 1, \ldots, k)$ is the maximum element on the kth level. However, if we denote $\alpha_k = \min(b_{i(k)}, i_p = 1, \ldots, N, p = 1, \ldots, k)$, then the problem of convergence of the BCF (1) under the condition that the series $\sum_{k=1}^{\infty} \alpha_k$ is divergent remains open (for more than 40 years).

In the proof of convergence of the BCF with positive elements, it is customary to use special inequalities for harmonic means. Their continual analogs were established by Mykhal'chuk and applied to the investigation of the convergence of integral continued fractions [3].

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Note that the inequality [1]

$$\sum_{j=1}^{n}\left(1+\delta\sum_{i=1}^{n}\frac{x_j}{x_i}+\gamma\sum_{i=1}^{n}\frac{y_j}{y_i}+\mu\left(\sum_{i=1}^{n}\frac{x_j}{x_i}\right)\left(\sum_{i=1}^{n}\frac{y_j}{y_i}\right)\right)^{-1} \le \left(\frac{1}{n}+\delta+\gamma+\mu\right)^{-1}, \tag{2}$$

where $\delta \ge 0$, $\gamma \ge 0$, $\mu \ge 0$, $x_i > 0$, $y_i > 0$, $i = 1,\ldots,n$, $n > 1$, used in the study of convergence of the BCF should be improved.

If, e.g., $x_1 \le x_2 \le \ldots \le x_n$ and $y_1 \ge y_2 \ge \ldots \ge y_n$, or vice versa, then the right-hand side in (2) can be re-placed with $\left(\frac{1}{n}+\delta+\gamma+n\mu\right)^{-1}$. However, if the sequences $\{x_i\}$ and $\{y_i\}$ are both monotonically increasing or monotonically decreasing, then inequality (2) should be replaced with the opposite inequality.

In the general case, it is possible to make inequality (2) sharper by using a more accurate upper bound for the function

$$f(z) = \frac{x_1 y_1 + x_2 y_2 + \ldots + x_n y_n}{(x_1 + x_2 + \ldots + x_n)(y_1 + y_2 + \ldots y_n)}, \quad z = (x_1,\ldots,x_n,y_1,\ldots,y_n).$$

Lemma 1. *For arbitrary nonnegative real numbers δ, γ, and μ and positive x, X $(x < X)$, y, Y $(y < Y)$, the following inequality holds:*

$$\sum_{j=1}^{n}\left(1+\delta\sum_{i=1}^{n}\frac{x_j}{x_i}+\gamma\sum_{i=1}^{n}\frac{y_j}{y_i}+\mu\left(\sum_{i=1}^{n}\frac{x_j}{x_i}\right)\left(\sum_{i=1}^{n}\frac{y_j}{y_i}\right)\right)^{-1} \le \left(\frac{1}{n}+\delta+\gamma+n_D\mu\right)^{-1} \tag{3}$$

under the condition that

$$z = (x_1,\ldots,x_n,y_1,\ldots,y_n) \in D, \quad n > 1,$$

where

$$D = \{0 < x \le x_i \le X, 0 < y \le y_i \le Y, i = 1,\ldots,n\}.$$

In this case,

$$n_D = \frac{((n-n_\varepsilon)X + n_\varepsilon x)((n-n_\varepsilon)Y + n_\varepsilon y)}{(n-n_\varepsilon)XY + n_\varepsilon xy}$$

and n_ε is the least integer satisfying the inequality

$$n_\varepsilon \ge \frac{n}{1-\varepsilon} - \frac{1}{2} - \frac{1}{2}\sqrt{1+\frac{4n^2\varepsilon}{1-\varepsilon^2}}, \quad \varepsilon = \frac{xy}{XY}.$$

Proof. We now find the highest value of the function $f(z)$ in D by assuming that $n > 1$. The points $z \in D$ such that $x_i = u$, $y_i = v$, $i = 1,\ldots,n$, and $x < u < X$, $y < v < Y$, are extreme. Since the quadratic form