Mathematical Model of Balanced Layout Problem Using Combinatorial Configurations

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Abstract: **The optimization of the balanced layout of a set of 3D-objects in a container is considered. We define combinatorial configurations describing the combinatorial structure of the problem. A mathematical model of the problem is presented. The model takes into account the placement constraints, the mechanical characteristics and the combinatorial features of the problem.**

Keywords: **Balanced Layout, 3D-objects, Combinatorial Configurations, Phi-function Technique.**

I. Introduction

Balanced layout problems belong to the class of NP-hard placement problems [1] and are the subject of the study of computational geometry, and the methods for their solution are a new branch of the theory of operations research [2, 3]. The essence of the problem lays in the search for the optimal placement of a given set of 3D-objects in a container, taking into account, so called, behavior constraints, which ensure the balance of the system under consideration [4], [5].

II. Problem Formulation

Let Ω be a container of height H that can take the form of a cuboid, cylinder, paraboloid of rotation, or truncated cone. The container Ω is defined in the global coordinate system $Oxyz$, where Oz is the longitudinal axis of symmetry. We assume that container Ω is divided by horizontal racks into sub-containers Ω^j, $j \in J_m = \{1, \dots, m\}$. We denote distances between racks S_j and S_{j+1} by t_j, $j \in J_m$, $\sum_{j=1}^{m} t_j = H$. The center of the lower base of container Ω is located in the origin of the global coordinate system $Oxyz$.

Let $A = \{\mathbb{T}_i, i \in J_n\}$ be a set of homogeneous 3D-objects given by their metrical characteristics. Each object $\mathbb{T}_i$ of height h_i and mass m_i, is defined in its local coordinate system $O_i x_i y_i z_i$, $i \in J_n$. The location of object $\mathbb{T}_i$ inside container Ω is defined by vector $u_i = (v_i, z_i, \theta_i)$, where (v_i, z_i) is a translation vector in the global coordinate system $Oxyz$, θ_i is a rotation angle of object $\mathbb{T}_i$ in the plane $O_i x_i y_i$, where $v_i = (x_i, y_i)$, and the value of z_i, $i \in J_n$, is uniquely defined by sub-container Ω^j, $j \in J_m$, in which object $\mathbb{T}_i$ will need to be placed.

In contrast to the BLP problems, where a priori the requirement for placing objects in specific sub-containers Ω^j, $j \in J_m$, is known, in this study the problem of the balanced layout of objects is formulated, which involves generation and selection of a partition of the set A into non-empty subsets A^j, $j \in J_m$. Here A^j is a subset of objects which have to be placed on rack S_j inside Ω^j.

On placement of object $\mathbb{T}_i$, $i \in J_n$, inside subcontainer Ω^j the following constraints are imposed

$$z_i = \sum_{l=1}^{j} t_{l-1} + h_i, \tag{1}$$

where $j \in J_m$. We consider that $t_0 = 0$ and $\forall i \in J_n$ there exists $j^* \in J_m$: $h_i \le t_{j^*}$.

Let $J_n^j \subseteq J_n$ be a set of indexes of objects which are placed in sub-container Ω^j, $j \in J_m$,

$$\bigcup_{j=1}^{m} J_n^j = J_n, \; J_n^i \cap J_n^j = \varnothing, \; i \ne j \in J_m; \tag{2}$$

$k_j = |A^j|$ is the number of objects which are placed in sub-container Ω^j, $k_j > 0$, $j \in J_m$,

$$\sum_{j=1}^{m} k_j = n. \tag{3}$$

In addition, the following placement constraints have to be taken into account:

$$\operatorname{int} \mathbb{T}_{i_1} \cap \operatorname{int} \mathbb{T}_{i_2} = \varnothing, \; i_1 < i_2 \in J_n^j, \; j \in J_m, \tag{4}$$

$$\mathbb{T}_i \subset \Omega^j, \; i \in J_n^j, \; j \in J_m, \tag{5}$$

$$h^j \le t_j, \; h^j = \max\{h_i^j, i \in J_n^j\}, \; j \in J_m. \tag{6}$$

We designate a system, formed as a result of the placement of objects $\mathbb{T}_i$ of the set A in container Ω by Ω_A, and a system of coordinates of Ω_A by $O_s XYZ$, where $O_s = (x_s(v), y_s(v), z_s(v))$ is the mass center of Ω_A

$$x_s(v) = \frac{\sum_{i=1}^{n} m_i x_i}{M}, \; y_s(v) = \frac{\sum_{i=1}^{n} m_i y_i}{M}, \; z_s = \frac{\sum_{i=1}^{n} m_i z_i}{M}, \quad (7)$$

$M = \sum_{i=1}^{n} m_i$ is the mass of system Ω_A and $O_s X \| Ox$, $O_s Y \| Oy$, $O_s Z \| Oz$.

We consider the deviation of the center of mass O_s of system Ω_A from given point (x_0, y_0, z_0) as an objective function.

Combinatorial Balanced layout Problem (CBLP). Define such variant of the partition of the object set A into nonempty subsets A^j, $j \in J_m$, and the corresponding placement parameters $u_i = (v_i, z_i, \theta_i)$ of objects $\mathbb{T}_i$, $i \in J_n$, taking into account relations (2)–(6), that the objective function will reach its minimum value.

We assume that the problem has at least one feasible solution.

Note. Restrictions on the axial and centrifugal moments of the system and allowable distances between objects may also be given.

III. Mathematical Model

Now we define special combinatorial configurations describing the discrete structure of the CBLP problem. Some basic approaches for mathematical modelling of optimization problems on combinatorial configurations are described in e.g., [6, 7].

The variants of partition of the set A into non-empty subsets A^j, $j \in J_m$, are determined by both the number of elements in each subset and the order of the subsets.

Let us consider the sub-containers and the assumed corresponding sets of objects A^j, $j \in J_m$. Then the tuple of natural numbers $(k_1, k_2, \ldots, k_m)$, such that $\sum_{j=1}^{m} k_j = n$, determines possible number k_j of objects in each sub-container Ω^j. The number of all such tuples is equal to the number of compositions of the number n of length m [8], which is $\left| C_{n-1}^{m-1} \right|$.

Let us derive in what ways it is possible to decompose n various objects from a set A into m sub-containers Ω^j, $j \in J_m$, if in sub-containers there are accordingly $k_1, k_2, \ldots, k_m$ objects, and sets of objects A^j, $j \in J_m$, inside corresponding sub-containers Ω^j, $j \in J_m$, are not ordered.

Without loss of generality, we will distinguish the objects with the same values of metrical characteristics, height h_i and mass m_i (for example, consider them to be different in number).

We order the elements of set A. To each object we assign the number of the sub-container into which it is expected to be placed. We get a tuple consisting of n elements that form a permutation with repetitions from m numbers $1, 2, \ldots, m$, in which the first element is repeated k_1 times, the second element is repeated k_2 times, ..., the last element is repeated k_m times.

The total number of permutations is equal to

$$P(n, k_1, k_2, \ldots, k_m) = \frac{n!}{k_1! \cdot k_2! \cdot \ldots \cdot k_m!}. \quad (8)$$

Then the number of variants of partitions of various objects from set A to m sub-containers Ω^j, provided that each sub-container contains at least one object and the order of placing objects inside the sub-container is not important, is equal to

$$\sum_{k_1+k_2+\ldots+k_m=n} P(n, k_1, k_2, \ldots, k_m) = \sum_{k_1+k_2+\ldots+k_m=n} \frac{n!}{k_1! \cdot k_2! \cdot \ldots \cdot k_m!}$$

Note that the number of summands in the sum is equal to $N = \left| C_{n-1}^{m-1} \right|$.

To generate subsets A^j, $j \in J_m$, we introduce a special combinatorial configuration [9].

Rather complex combinatorial configurations can formally be described and generated using tools of construction of compositional κ-images of combinatorial sets (κ-sets) proposed in [10]. A combinatorial set is considered as a set of tuples that constructed from a finite set of arbitrary elements (so-called generating elements) according to certain rules. Permutations, combinations, arrangements, and binary sequences are the examples of classical combinatorial sets.

The basic idea of generation of κ-sets is introduced in [10]. However, the problem of generating κ-sets of more complicated combinatorial structure remains the open problem. Just one of such special cases is studied in [11].

The problem of generating κ-sets is based on special techniques of generating base combinatorial sets. The base sets can be defined as combinatorial sets with the known descriptions: both classical combinatorial sets (permutations, combinations, arrangements, tuples) or non-classical combinatorial sets (permutations of tuples, compositions of permutations, permutations with a prescribed number of cycles, etc.). Generation algorithms for some of base combinatorial sets are presented in, e.g., [12-15].

We denote $\mathbb{C}_{\mathbb{P}}(n, m)$ the set of compositions of the number n of length m (which corresponds to the partition of different objects from set A to m sub-containers Ω^j, $j \in J_m$, provided that each sub-container contains at least one object and the order of objects inside the sub-container is not important). Wherein, $\left| \mathbb{C}_{\mathbb{P}}(n, m) \right| = N = \left| C_{n-1}^{m-1} \right|$.

Let $\mathbb{k} = (k_1, \ldots, k_m) \in \mathbb{C}_{\mathbb{P}}(n, m)$, $\sum_{j=1}^{m} k_j = n$, $k_j \geq 1$, $j \in J_m$.

We introduce a combinatorial set $\mathbb{Q}(\mathbb{k})$ that is a composition image of combinatorial sets (κ-set) $\mathbb{C}_{\mathbb{P}}(n, m)$; $C_n^{k_1}$, $C_{n_1}^{k_2}$, $C_{n_2}^{k_3}$, …, $C_{n_{m-1}}^{k_m}$, generated by sets

I_{n_0}, I_{n_1}, I_{n_2}, …, $I_{n_{m-1}}$, where $n_i = n - k_1 - \ldots - k_i$, $i \in J_{m-1}$, $I_{n_0} = J_n$,

$$I_{n_1} = I_{n_0} \setminus \{j_1^{n_0}, j_2^{n_0}, \ldots, j_{k_1}^{n_0}\},\ (j_1^{n_0}, j_2^{n_0}, \ldots, j_{k_1}^{n_0}) \in C_n^{k_1},$$

$$I_{n_2} = I_{n_1} \setminus \{j_1^{n_1}, j_2^{n_1}, \ldots, j_{k_2}^{n_1}\},\ (j_1^{n_1}, j_2^{n_1}, \ldots, j_{k_2}^{n_1}) \in C_{n_1}^{k_2},$$

$$\ldots$$

$$I_{n_{m-1}} = I_{n_{m-2}} \setminus \{j_1^{n_{m-2}}, j_2^{n_{m-2}}, \ldots, j_{k_{m-1}}^{n_{m-2}}\},$$
$$(j_1^{n_{m-2}}, j_2^{n_{m-2}}, \ldots, j_{k_{m-1}}^{n_{m-2}}) \in C_{n_{m-2}}^{k_{m-1}},$$

$$I_{n_{m-1}} = \{j_1^{n_{m-1}}, j_2^{n_{m-1}}, \ldots, j_{k_m}^{n_{m-1}}\},$$
$$(j_1^{n_{m-1}}, j_2^{n_{m-1}}, \ldots, j_{k_m}^{n_{m-1}}) \in C_{n_{m-1}}^{k_m}.$$

Note that

$$I_{n_0} \cup I_{n_1} \cup \ldots \cup I_{n_{m-1}} = J_n = \{1, 2, \ldots, n\},$$
$$I_{n_s} \cap I_{n_t} = \varnothing,\ s \neq t \in J_{m-1}^0 = \{0, 1, \ldots, m-1\}.$$

Each element $q(\Bbbk) \in \mathcal{Q}(\Bbbk)$ can be described in the form

$$q(\Bbbk) = (q_1, \ldots, q_{k_1} \left| q_{k_1+1}, \ldots, q_{k_1+k_2} \right|, \ldots,$$
$$\left| q_{k_1+\ldots+k_{m-1}}, \ldots, q_{k_{m-1}+k_m} \right),$$

where $(q_1, \ldots, q_{k_1}) = (j_1^{n_0}, j_2^{n_0}, \ldots, j_{k_1}^{n_0}) \in C_n^{k_1}$,

$$(q_{k_1+1}, \ldots, q_{k_1+k_2}) = (j_1^{n_1}, j_2^{n_1}, \ldots, j_{k_2}^{n_1}) \in C_{n_1}^{k_2},$$
$$\ldots$$
$$(q_{k_1+\ldots+k_{m-1}}, \ldots, q_{k_{m-1}+k_m}) = (j_1^{n_{m-1}}, j_2^{n_{m-1}}, \ldots, j_{k_m}^{n_{m-1}}) \in C_{n_{m-1}}^{k_m}.$$

The cardinality of set $\mathcal{Q}(\Bbbk)$ is derived by (9).

An element $q(\Bbbk)$ of the set $\mathcal{Q}(\Bbbk)$ is said to be *a tuple of partition* of the set *A* into subsets A^j, $j \in J_m$.

Now we define the vector of the basic variables of the problem CBLP: $u = (v, z, \theta)$, where $v = (v_1, \ldots, v_n) \in \mathbf{R}^{2n}$, $\theta = (\theta_1, \ldots, \theta_n) \in \mathbf{R}^n$, $v_i = (x_i, y_i) \in \mathbf{R}^2$, x_i, y_i, θ_i are continuous variables, $z = (z_1, \ldots, z_n) \in \mathbf{R}^n$, z_i, $i \in J_n$, are discrete variables defined by (1).

The values of variables z_i, $i \in J_n$, are determined in the order given by elements $q(\Bbbk)$ of combinatorial set $\mathcal{Q}(\Bbbk)$:

$$z_{q_i} = \sum_{l=1}^{s} t_{l-1} + h_{q_i}, \tag{10}$$

where

$$s = \begin{cases} 1, \text{ if } & i \leq k_1, \\ 2, \text{ if } & k_1 < i \leq k_1 + k_2, \\ \ldots & \\ m, \text{ if } & k_1 + k_2 + \ldots + k_{m-1} < i \leq k_1 + k_2 + \ldots + k_m, \end{cases}$$

$i = 1, 2, .., n$, $q_i \in \{1, 2, .., n\}$, $q(\Bbbk) \in \mathcal{Q}(\Bbbk)$.

Let us formalize placement constraints (4)-(6), using phi-function technique.

We consider two objects $\mathbb{T}_1$ and $\mathbb{T}_2$ with variable parameters $u_1 = (v_1, z_1, \theta_1) \in \mathbf{R}^3$, $u_2 = (v_2, z_2, \theta_2) \in \mathbf{R}^3$, where $v_1 = (x_1, y_1)$, $v_2 = (x_2, y_2)$, $x_1, y_1, \theta_1\ x_2, y_2, \theta_2$ are continuous variables and z_1, z_2 are discrete variables.

By definition [2, 3] a phi-function is a continuous function, therefore we extend the concept to discrete variables z_1, z_2.

Definition 1. Function $\Upsilon_{12}(u_1, u_2)$ is called a *D-phi-*function of 3D-objects $\mathbb{T}_1$ and $\mathbb{T}_2$ if function $\Upsilon_{12}(v_1, z_1^0, \theta_1, v_2, z_2^0, \theta_2)$ is a *phi*-function $\Phi_{12}(v_1, z_1^0, \theta_1, v_2, z_2^0, \theta_2)$ of objects $\mathbb{T}_1$ and $\mathbb{T}_2$ for fixed values $z_1 = z_1^0$ and $z_2 = z_2^0$.

Definition 2. Function $\Upsilon'_{12}(u_1, u_2, u_{12})$ is called a quasi D-*phi*-function of 3D-objects, $\mathbb{T}_1$ and $\mathbb{T}_2$ if function $\Upsilon'_{12}(v_1, z_1^0, \theta_1, v_2, z_2^0, \theta_2, u_{12})$ is a quasi-*phi*-function $\Phi'_{12}(v_1, z_1^0, \theta_1, v_2, z_2^0, \theta_2, u_{12})$ of objects $\mathbb{T}_1$ and $\mathbb{T}_2$ for fixed values $z_1 = z_1^0$ and $z_2 = z_2^0$.

Here u_{12} is the vector of auxiliary continuous variables that is used to constructs a quasi *phi*-function of two objects $\mathbb{T}_1$ and $\mathbb{T}_2$.

The placement constraints (4)-(6) are described by the system of inequalities $\Upsilon_1(u, \tau) \geq 0$, $\Upsilon_2^*(u) \geq 0$, where the inequality $\Upsilon_1(u, \tau) \geq 0$ describes the non-overlapping constraints and the inequality $\Upsilon_2^*(u) \geq 0$ describes the containment constraints

$$\Upsilon_1(u, \tau) = \min\{\Upsilon_1^j(u, \tau),\ j \in J_m\},$$
$$\Upsilon_1^j(u, \tau) = \min\{\Upsilon_{q_1 q_2}^j(u_{q_1}, u_{q_2}, u_{q_1 q_2}),\ q_1 < q_2 \in J_n^j\}, \tag{11}$$
$$\tau = (u_{q_1 q_2},\ q_1 < q_2 \in J_n^j),\ \Upsilon_2^*(u) = \min\{\Upsilon_2^{*j}(u),\ j \in J_m\},$$
$$\Upsilon_2^{*j}(u) = \min\{\Upsilon_{q_i}^*(u_{q_i}),\ q_i \in J_n^j\}, \tag{12}$$

$\Upsilon_{q_1 q_2}^j(u_{q_1}, u_{q_2}, u_{q_1 q_2})$ is the function that describes non-overlapping of objects $\mathbb{T}_{q_1}$ and $\mathbb{T}_{q_2}$, $u_{q_1} = (x_{q_1}, y_{q_1}, z_{q_1}, \theta_{q_1})$, $u_{q_2} = (x_{q_2}, y_{q_2}, z_{q_2}, \theta_{q_2})$,

$\Upsilon^*_{q_i}(u_{q_i})$ is the function that describes non-overlapping of objects $\mathbb{T}_{q_i}$ and $\Omega^{*j} = \mathbf{R}^3 / \operatorname{int}\Omega^j$.

Thus, in relations (11), (12) for fixed values z_{q_1} and z_{q_2}, we have: $\Upsilon^j_{q_1 q_2}(u_{q_1}, u_{q_2})$ is a *phi*-function [16] $\Phi^{\mathbb{T}\mathbb{T}}_{q_1 q_2}(u_{q_1}, u_{q_2})$ for objects $\mathbb{T}_{q_1}$ and $\mathbb{T}_{q_2}$ or a quasi-*phi*-function [17] $\Phi'^{\mathbb{T}\mathbb{T}}_{q_1 q_2}(u_{q_1}, u_{q_2}, u_{q_1 q_2})$ for objects $\mathbb{T}_{q_1}$ and $\mathbb{T}_{q_2}$; $\Upsilon^*_{q_i}(u_{q_i})$ is a *phi*-function $\Phi^{\mathbb{T}\Omega^{*j}}_{q_i}(u_{q_i})$ for objects $\mathbb{T}_{q_i}$ and Ω^{*j}.

If the minimum allowable distances between objects are given, adjusted *phi*-functions (quasi-*phi*-functions) are used for the corresponding pairs of objects [16, 17].

Mathematical model of the CBLP problem can be defined as follows:

$$F(u^*, \tau^*) = \min F(u, \tau) \text{ s.t. } (u, \tau) \in W, \tag{13}$$

$$W = \{(u, \tau) \in \mathbf{R}^\sigma : \Upsilon_1(u, \tau) \ge 0, \Upsilon^*_2(u) \ge 0, \mu(u) \ge 0\}, \tag{14}$$

where

$$F(u) = d = (x_s(v,z) - x_0)^2 + (y_s(v,z) - y_0)^2 + (z_s - z_0)^2$$

$$u = (v, z, \theta), \quad v = (v_1, \dots, v_n), \quad \theta = (\theta_1, \dots, \theta_n),$$

$$v_i = (x_i, y_i), \quad i \in J_n, \quad z = (z_1, \dots, z_n),$$

function $\Upsilon_1(u, \tau)$ is described by (11) with $\Xi = \bigcup_{j=1}^{m} \Xi^j$, $\Xi^j = \{(q_1, q_2) : q_1 < q_2 \in J^j_n\}$, $\tau = (\tau_1, \dots, \tau_s) = (u_{q_1 q_2}, q_1 < q_2 \in J^j_n)$ is a vector of auxiliary variables for quasi phi-functions, $s = |\Xi|$, function $\Upsilon^*_2(u)$ is defined by (12), elements of vector z are given by (10), $\mu(u) \ge 0$ describes behavior constraints.

CBLP problem can be represented as a mixed integer programming (MIP) problem, using approach with boolean variables.

However, unlike (13)-(14), this approach leads to increasing the number of discrete variables of the model and therefore increases the dimension of the CBLP problem in MIP form.

IV. Conclusion

We study the problem of the balanced layout of 3D-objects within a container divided by horizontal racks onto sub-containers.

A mathematical model has been constructed that takes into account not only the geometrical and behavior constraints, but also combinatorial features of the problem associated with the construction of partitions of the set of placed objects into sub-containers.

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