JOURNAL
OF EUROPEAN ECONOMY
Vol. 11. Special issue – 2012
Publication of Ternopil National Economic University

Kenji MIYAZAKI

A HUMP-SHAPED RELATIONSHIP BETWEEN INFLATION AND ENDOGENOUS GROWTH

Abstract

This study explores the relationship between inflation and economic growth using a transaction costs model with a socially determined discount rate and a linear production technology. Even when the labor decision is inelastic, this study demonstrates that inflation affects balanced growth path (BGP) with nonconstant time preferences. In particular, if the degree of impatience increases in the economy-wide average ratio of general assets (a weighted sum of capital and money) to consumption, then a certain rate of money supply can achieve maximized endogenous growth. The numerical examples demonstrate a hump-shaped relationship between inflation and BGP, but the impact is quantitatively small.

Key words:

Monetary policy, impatience, hump-shaped.

JEL: E5, O42.

Miyazaki Kenji, Professor, Faculty of Economics, Hosei University.

1. Introduction

Empirical evidence suggests that inflation and economic growth have a hump-shaped relationship. Gomme (1993) and Bullard and Keating (1995) used international cross-country data to show a positive correlation between inflation and growth rate in low inflation countries, whereas there is a negative correlation link in high inflation countries. Ahmed and Rogers (2003) utilized long-term US time series data to demonstrate that moderate inflation has a positive effect on output, whereas an unusually high rate of inflation has a negative effect. The purpose of this paper is to build a simple monetary model generating a hump-shaped relationship between inflation and economic growth.

Beginning with Tobin (1965) and Sidrauski (1967), the relationship between inflation and economic growth has been one of the central issues in monetary economic theory. In an exogenous growth framework, Wang and Yip (1992) considered the money-in-the-utility-function (MIUF) approach, the cash-in-advance (CIA) approach, and the transaction costs (TC) approach, and demonstrated that when labor decision is inelastic, these three approaches lead to superneutrality, or no effect of monetary expansion on real variables such as capital stock and consumption.[1] When labor supply is elastic, an increasing money supply generally lowers the level of capital stocks in the long run. In an exogenous growth framework, rates of inflation and money supply are the same in a steady state, and economic growth is interpreted as the level of capital stocks.

Several authors have adopted an endogenous growth framework in which economic growth is interpreted as the rate on the balanced growth path (BGP), and the rate of inflation is the difference between the rates of monetary expansion and BGP. In the CIA approach, Gomme (1993) introduced a trade-off between labor and consumption, and Jones and Manuellir (1995) considered nominal rigidities. In the TC approach, De Gregorio (1993) and Jha et al. (2002) used a transaction technology in which the transaction cost function increases with consumption and decreases with money in De Gregorio (1993), and additionally increases in real output in Jha et al. (2002). As in an exogenous framework, these studies cannot explain a positive relationship between monetary expansion and BGP.

For a positive relationship in an endogenous framework, Fukuda (1994) and Itaya and Mino (2003) considered a monetary version of the Benhabib and Farmer (1994) model. Fukuda and Itaya and Mino adopted the CIA and the TC approaches, respectively. Both studies found that with sufficiently large labor externality, two equilibriums emerge, of which one has a positive relationship be-

[1] To be precise, the cash-in-advance constraint should be imposed only on consumption, and the transaction costs should be a function of consumption and money.

tween monetary expansion and the BGP[2]. However, these models cannot explain a hump-shaped relationship.

The studies cited so far, which fail to predict a hump-shaped relationship between inflation and economic growth, have assumed the discount rate to be constant. A constant discount rate is adopted in standard macroeconomic models just for simplicity. Empirical studies including that of Becker and Mulligan (1997)[3] have reported that the discounting rate is not constant and depends on other economic variables. In an endogenous growth framework, a positive growth rate results from the difference between the marginal productivity of capital and time preference. Therefore, the varying discount rate is not only more consistent with empirical evidence than a constant rate but is also an important candidate to explain a hump-shaped relationship[4].

This study explores the relationship between inflation and economic growth using the TC model with a socially determined discount rate and a linear production technology. Even when the labor decision is inelastic, this study demonstrates that inflation affects endogenous growth with nonconstant time preferences. In particular, if the degree of impatience increases in the economy-wide average ratio of general assets (a weighted sum of money and capital) to consumption, then the relationship between inflation and economic growth may be hump-shaped, which supports empirical findings. In the case of total assets (the sum of capital and money), no monetary expansion achieves maximized economic growth. We also establish the existence and uniqueness of a BGP and demonstrate that such a BGP is locally stable.

The intuition is as follows. The higher the rates of money supply, the lower are real balances and the higher are transaction costs, and this discourages consumption, or increases the ratio of capital to consumption. At the same time, a higher cost of money holdings decreases the money demand or the ratio of money to consumption. When an economic agent is less patient as the ratio of assets to consumption increases, the agent becomes either less patient if the capital effect is stronger or more patient ifthe money effect is stronger. When the cost of holding money is sufficiently high, the capital effect is dominant, lowering the rate of economic growth.

Furthermore, we conduct several numerical exercises and compare our model with one in which the discount rates are determined internally by the individual. Our numerical exercises confirm that the hump-shaped relationship be-

---

[2] Both also showed that one equilibrium is indeterminate, implying sunspot equilibriums. In Itaya and Mino, the convexity of the transaction cost function plays an important role in determining which of two equilibriums is indeterminate and has a positive relationship between monetary expansion and the BGP.

[3] Chen et al. (2008) referred to other empirical papers with nonconstant time preference.

[4] One explanation for this hump-shaped relationship with the same technology and preferences was provided by Dutta and Kapur (1998), who considered a three-period overlapping model with preference shocks, CIA constraints, irreversible investment, and exogenous technology.

tween inflation and the rate of economic growth is established with a set of plausible parameters, but show that the impact of inflation on economic growth is quantitatively small. Our comparison proves that even if the discount rate depends on not aggregate but individual consumption, then the results of the comparative statics of the BGP are observationally equivalent in both models.

This study is not the first attempt to consider the relationship between money and growth with varying discounting rates. Kam (2005) considered the MIUF model with discount rates depending on economy-wide total assets and found that when discount rates increase in total assets, then a positive relationship between inflation and capital stocks, a Tobin effect, emerges. Chen et al. (2008) used the MIUF and TC models where the degree of impatience was internally determined by individuals. Chen et al. demonstrated that increasing impatience in money (resp. consumption) leads to a Tobin effect in the MIUF (resp. TC) model. However, both studies lay in an exogenous growth framework and did not generate a hump-shaped relationship.

This study is also not the first attempt to use an endogenous growth model with varying discount rates. Palivos et al. (1997) and Meng (2006) investigated the relationship between the BGP and functional forms of the felicity and discount rates in the framework of linear technology. Both studies found that under the BGP and the discount rate function invariant to the BGP, the discount rate should be constant or a homogeneous degree of zero, and that the elasticity of marginal felicity must be constant. We demonstrate that their results hold in our monetary economy.

Our study makes three contributions. First, our model provides a solution to the puzzle posed by Gomme (1993), Bullard and Keating (1995) and Ahmed and Rogers (2003), and sheds new light on monetary policy. Second, we consider the monetary model with varying discount rate in an endogenous growth framework, whereas Palivos et al. (1997) and Meng (2006) focused their attention on a real economy. Third, we find that in some types of models, the comparative statics of the BGP have the same results whether consumption in the discount rate is determined externally or internally.

Our paper is organized as follows. Section 2 describes a model economy with externally determined discount rates, and Section 3 establishes the existence and uniqueness of the BGP. Section 4 presents a comparative statics analysis, and Section 5 discusses local stability. Section 6 presents several numerical exercises, and Section 7 compares our model with that with discount rates determined internally by the individual. Section 8 concludes.

2. The Model

We construct an endogenous growth model augmented with real money balances or simply money. No uncertainty exists. The representative agent with perfect foresight is infinitely lived and endowed with initial capital $k_0 > 0$, an initial nominal money stock $\bar{m}_0 > 0$, and a normalized initial price level $P_0 = 1$. The production technology is linear in capital, whereas the labor supply is inelastic. The population of the economy stays constant.

The representative agent has the following lifetime utility:

$$U = \int_0^{\infty} u(c_t) e^{-\Delta_t} dt, \tag{1}$$

where c_t is consumption at t , u represents felicity, and Δ_t represents the cumulative discount rate at period t .

The felicity function u is continuous, increasing, and concave, as usual. The cumulative discount rate function Δ_t is determined by:

$$\dot{\Delta}_t = \rho(C_t, X_t), \tag{2}$$

where ρ represents the discount rate function, C_t is the economy-wide average level of consumption at t , and X_t is the economy-wide average level of assets at t . The discounting rate for each agent is not necessarily constant, but is taken as exogenous. Society determines the time preference, and individuals accept a social norm to maximize their preference. We assume $\Delta_0 = 0$ and $0 < \rho < \infty$ for all C and X . Other assumptions are described by the functions u and ρ after we present some propositions.

Equation (2) indicates that not only aggregate consumption but also assets affect degree of impatience. We consider two types of assets: productive and nonproductive. Economy-wide productive assets, denoted by K , contribute directly to production at a macroeconomic level, whereas nonproductive assets, denoted by M , facilitate transactions. Below, we call K and M capital and money respectively, and refer to the sum of capital and money $K + M$ as total assets. As for the variable X in (2), we can consider the economy-wide general assets:

$$X = \alpha_1 K + \alpha_2 M,$$

where $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$. When $\alpha_1 = 1$ and $\alpha_2 = 1$, then X represents total assets.

Individual money or real balance, denoted by m , is introduced to the model by considering the costs of individual transactions. The amount of transaction costs increases with consumption, but decreases with real money balances. Such transaction cost technology is represented by $T(c, m)$. For the existence of a BGP, we assume $T = s\left(\frac{m}{c}\right)c$, $s \geq 0$, $s' < 0$, $s'' > 0$ for all $\frac{m}{c} > 0$, $\lim_{\frac{m}{c}\to\infty} s'\left(\frac{m}{c}\right) = 0$, and $\lim_{\frac{m}{c}\to 0} s'\left(\frac{m}{c}\right) = -\infty$. This assumption ensures $T_c > 0$, $T_m < 0$, $T_{cc} > 0$, $T_{mm} > 0$, and $T_{cc}T_{mm} - T_{cm}^2 = 0$. We call $s\left(\frac{m}{c}\right)$ the TC (per unit of consumption) function. An example is:

$$s\left(\frac{m}{c}\right) = s_0\left(\frac{m}{c}\right)^{-\eta}$$

where $s_0 > 0$ and $\eta > 0$.

The budget constraint is:

$$\left[1 + s\left(\frac{m}{c}\right)\right]c + \dot{a} = Ak + v - \pi m, \tag{3}$$

where $a = m + k$ is individual total assets, k is individual capital, $\pi = \frac{\dot{P}}{P}$ is expected rates of inflation, v is lump-sum transfers from the government, and A is a constant parameter representing a linear technology of production function[5].

Given C and X , the economic agent chooses c , k , and m to maximize (1) subject to (2), (3), and the boundary conditions $k_0 > 0$, $\bar{m}_0 > 0$, $P_0 = 1$, and $\lim \lambda(k + m)e^{-\Delta} = 0$. Because the utility function is bounded, the optimization problem is well defined. When λ is the costate variable of (3), Pontryagin's maximum principle yields:

$$\lambda = \frac{u'}{1 + s - \frac{s'm}{c}} \tag{4}$$

$$\lambda(A + \pi + s') = 0 \tag{5}$$

$$\dot{\lambda} = \lambda(\rho - A). \tag{6}$$

The government behaves in a conventional way (according to monetary theory). It prints nominal money at a constant rate μ and runs a balanced budget by transferring seigniorage revenues to consumers in a lump-sum manner: $v_t = \mu m_t$. In Section 4, we discuss the optimal monetary policy, which depends on reaction from consumers. For the optimal policy, we assume that the government can commit to future rates of monetary growth.

In equilibrium, the money and the goods markets are clear:

$$\dot{m} = (\mu - \pi)m \tag{7}$$

[5] Because the individual level of total assets is denoted by a , the economy-wide average level of total assets should have been A . However, the notation A has been conventionally used to represent a constant technology of production function and we follow this convention. Thus, as the average level of total assets, we use $K + M$.

$$\dot{k} = Ak - (1+s)c. \tag{8}$$

The aggregate consistency condition requires $c = C$, $k = K$, and $m = M$. A monetary equilibrium is a set of paths $\{c_t, k_t, m_t, \pi_t\}_{t\in[0,\infty)}$ that maximizes (1) subject to (2) and (3) for given initial conditions, in which the government behavior condition, the market equilibrium conditions, and the aggregate consistency condition hold. We can obtain the following dynamic system of c, m, and k under a monetary equilibrium:

$$\theta\frac{\dot{c}}{c} = A - \rho + \frac{s'' \cdot \frac{m}{c^2}}{1 + s - s' \cdot \frac{m}{c}}\dot{m}, \tag{9}$$

$$\frac{\dot{m}}{m} = \mu + A + s', \tag{10}$$

and (8) with the boundary conditions, where

$$\theta = -c\frac{u''}{u'} + \frac{s'' \cdot \frac{m^2}{c^2}}{1 + s - s' \cdot \frac{m}{c}}, \tag{11}$$

The rate of inflation π is not included in the above dynamic system, but is determined by $\pi = \mu - \frac{\dot{m}}{m} = -A - s'$.

3. A Balanced Growth Path

In this section, we establish the existence and uniqueness of a BGP. A nondegenerate BGP monetary equilibrium is a set of monetary equilibrium paths $\{c_t, k_t, m_t, \pi_t\}_{t\in[0,\infty)}$ such that the quantity variables c, k, and m grow at a constant rate $\gamma > 0$. On the BGP,

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k} = \frac{\dot{m}}{m} = \gamma$$

and a constant $\pi = \mu - \gamma$. Note that the rate of nominal interests is $A + \pi = A + \mu - \gamma = -s'$ from (10). For a positive nominal interest rate, the government should set the growth of money at $\mu \geq \gamma - A$ on the BGP.

As Meng (2006) has shown, we can easily prove the following proposition.

Proposition 1: If a BGP exists and the discount rate function $\rho(C, X)$ is invariant to the BGP, then (i) $\rho(C, X)$ must be constant or homogeneous of degree zero in C and X, and (ii) the elasticity of marginal felicity must be constant.

Proof: The proof is essentially the same as that of Lemma 4.1 of Meng (2006), and is omitted.

In what follows, we assume $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ for $\sigma > 1$, and $\rho(C, X) = \rho\left(\frac{X}{C}\right)$. Note that the objective function (1) is well-defined in this utility case because $\frac{c^{1-\sigma}}{1-\sigma} < 0$ for $\sigma > 1$.

The discounting rate $\rho\left(\frac{X}{C}\right)$ depends on the ratio of economy-wide assets to consumption. When $\rho^{'} = 0$, the time preference rate is constant. When $\rho^{'} > 0$, the degree of impatience decreases with the economy-wide average level of consumption, but increases with the economy-wide average level of capital, money, or total assets, given that all other variables are fixed. For instance, $\rho^{'}\left(\frac{X}{C}\right) > 0$ indicates that as society becomes wealthier or consumes less, people become more impatient or less willing to defer consumption. Meng (2006) has examined the ratio of consumption to income in a real economy. The ratio is the same as the inverse ratio of capital to consumption in our framework because the production function is Ak .

Let $Z_1 = \frac{k}{c}$ and $Z_2 = \frac{m}{c}$. On the BGP, (8), (9), and (10) are expressed as

$$\sigma\gamma = A - \rho(\xi), \tag{12}$$

$$\gamma = \mu + A + s^{'}(Z_2), \tag{13}$$

$$\gamma = A - \frac{\{1 + s(Z_2)\}}{Z_1}, \tag{14}$$

where $\xi = \alpha_1 Z_1 + \alpha_2 Z_2$. Note that as long as the government sets the growth rate of the money supply to be $\mu \geq \frac{(1-\sigma)A}{\sigma}$, the nominal interest rate is always positive. In fact, $A + \mu - \gamma = \mu + \frac{(\sigma - 1)A}{\sigma} + \frac{\rho}{\sigma} > 0$ for all $\mu \geq \frac{(1-\sigma)A}{\sigma}$.

With additional assumptions regarding ρ , we can prove that there exists a unique BGP.

Proposition 2: Assume $\rho(\xi) > 0$ for all $\xi = \alpha_1 Z_1 + \alpha_2 Z_2 > 0$, and $\sup \rho(\xi) \equiv \bar{\rho} < A$. Then there exists a nondegenerate BGP for any $\mu \geq \frac{(1-\sigma)A}{\sigma}$ with $\sigma > 1$. When ρ is constant, or an increasing function only of Z_2 ($\rho^{'}(\xi) \geq 0$ and $\alpha_1 = 0$), then the BGP is unique. Even in the case with $\alpha_1 > 0$, there exists a unique BGP when $\rho^{'} \geq 0$ for all $\xi > 0$ and $(1+s)s^{''} - s^{'}(\mu + s^{'}) > 0$ for all $Z_2 > 0$.

Proof: See the Appendix.

In what follows, we assume that $\rho(\xi) > 0$, $\rho'(\xi) \geq 0$, and $\sup \rho(\xi) \equiv \bar{\rho} < A$. When $\alpha_1 > 0$, we need an additional assumption, $(1+s)s'' - s'(\mu + s') > 0$ for a unique BGP. This assumption is satisfied when $\mu \geq 0$ and $s(Z_2) = s_0 Z_2^{-\eta}$ where $s_0 > 0$ and $\eta > 0$ [6]. This functional specification is used in Section 6, where the uniqueness of the BGP is numerically confirmed in the case where $\mu < 0$.

4. Comparative Statics

In this section, we investigate the effect of monetary policy on the BGP. Our model allows the government to control the growth rate of money. We conduct a comparative static analysis of the BGP. We examine the effect of monetary expansion on economic growth, the ratio of capital to consumption, and the ratio of money to consumption. Then we demonstrate that there exists a monetary policy maximizing economic growth. We also assess the effect of monetary expansion on welfare, and investigate whether there exists a monetary policy to maximize welfare. We finally discuss the relationship between economic growth and the rate of inflation.

We perform the comparative statics with respect to γ, Z_1, and Z_2 in response to the money growth rate μ. The total differentiation of (12), (13), and (14) leads to the comparative-static results with respect to γ, Z_1, and Z_2 in response to the growth rate of money[7]

$$\begin{bmatrix} \dfrac{d\gamma}{d\mu} \\ \dfrac{dZ_1}{d\mu} \\ \dfrac{dZ_2}{d\mu} \end{bmatrix} = \frac{1}{\Omega} \begin{bmatrix} -\rho' \left\{ \dfrac{\alpha_1 s'(Z_2)}{Z_1} + \dfrac{\alpha_2 \left(1 + s(Z_2)\right)}{Z_1^2} \right\} \\ \dfrac{\sigma s'(Z_2)}{Z_1} - \alpha_2 \rho' \\ \alpha_1 \rho' + \dfrac{\sigma\left(1 + s(Z_2)\right)}{Z_1^2} \end{bmatrix}$$

where

$$\Omega = -\frac{\sigma s''(1+s)}{Z_1^2} - \alpha_1 \rho' \left(s'' + \frac{s'}{Z_1} \right) - \frac{\alpha_2 \rho' (1+s)}{Z_1^2}.$$

We have assumed $\rho' \geq 0$. Thus, the term Ω is negative when $s''(Z_2) + \dfrac{s'(Z_2)}{Z_1} > 0$. The condition $s''(Z_2) + \dfrac{s'(Z_2)}{Z_1} > 0$ is rewritten as

---

[6] In fact, $(1+s)s'' - s'(\mu + s') \geq (1+s)s'' - (s')^2$ for all $Z_2 > 0$ and $\mu \geq 0$.
[7] The detailed derivation is available by a request from the author.

$$\frac{m}{k}=\frac{Z_2}{Z_1}<-\frac{Z_2 s''}{s'}.$$

This condition is more likely to hold if the degree of concavity of s is sufficiently strong or if the amount of money is less than capital on the BGP. When $s\left(\frac{m}{c}\right)=s_0\left(\frac{m}{c}\right)^{-\eta}$ for example, then $-\frac{Z_2 s''(Z_2)}{s'(Z_2)}=\eta+1$. We assume that $\Omega<0$ within this section.

When $\Omega<0$ and $\rho^{'}\geq 0$, the monetary expansion raises the ratio of capital to consumption ($\frac{dZ_1}{d\mu}>0$) and lowers the ratio of money to consumption ($\frac{dZ_2}{d\mu}<0$). The effect of monetary expansion on economic growth is

$$\frac{d\gamma}{d\mu}=-\rho^{'}\frac{\alpha_1 s^{'}(Z_2)Z_1+\alpha_2\left(1+s(Z_2)\right)}{Z_1^2\Omega}. \tag{15}$$

Note that economic growth is independent of the rate of money supply when either $\rho^{'}=0$ or $\alpha_1=\alpha_2=0$. Except for these cases, the sign depends on the size of α_1 and α_2. For further analysis, we consider two extreme cases: $\xi=Z_1$ ($\alpha_1=1$ and $\alpha_2=0$) and $\xi=Z_2$ ($\alpha_1=0$ and $\alpha_2=1$), and then return to general cases.

With $\alpha_1=1$ and $\alpha_2=0$, the effect of monetary expansion on economic growth is $\frac{d\gamma}{d\mu}=-\frac{\rho' s'}{Z_1\Omega}$. When the degree of impatience is an increasing function of the ratio of capital to consumption ($\rho^{'}\left(\frac{K}{C}\right)>0$), the effect of growth rates of money has a negative effect on economic growth. The intuition is as follows: higher rates of money supply lower real balances, raise transaction costs, and therefore discourage consumption. When the degree of impatience increases in the ratio of capital to consumption, decreasing consumption makes the agent less patient, and thus lowers economic growth $\frac{A-\rho}{\sigma}$.

With $\alpha_1=0$ and $\alpha_2=1$, the effect of monetary expansion on economic growth is reversed ($\frac{d\gamma}{d\mu}=-\frac{\rho'(1+s)}{Z_1^2\Omega}$). When the degree of impatience increases in the ratio of real balances to consumption ($\rho^{'}\left(\frac{M}{C}\right)>0$), higher money rates have a positive effect on economic growth. Intuitively, given a common growth rate γ, the higher cost of holding money decreases the ratio of money to consumption Z_2 from (13). When an economic agent is more impatient as the ratio of money to consumption increases, decreasing Z_2 makes the agent more patient, and thus raises economic growth $\frac{A-\rho}{\sigma}$.

In general ($\alpha_1 > 0$ and $\alpha_2 > 0$), both the effects in the cases of $\rho'\left(\frac{K}{C}\right) > 0$ and $\rho'\left(\frac{M}{C}\right) > 0$ are mixed. The higher rates of money supply lower real balances, raise transaction costs, and therefore discourage consumption by increasing Z_1. At the same time, a higher cost of money holdings decreases the ratio of money to consumption Z_2 from (13). When an economic agent is less patient as $\frac{X}{C}$ increases ($\rho' > 0$), increasing Z_1 and decreasing Z_2 makes the agent *either* less patient if the capital effect is stronger *or* more patient if the money effect is stronger. Thus the effect on economic growth depends on which effect is dominant.

Recall that $\lim_{\frac{m}{c}\to 0} s'\left(\frac{m}{c}\right) = -\infty$ by assumption. Thus, a sufficiently high growth rate of money makes $\alpha_1 s'(Z_2)Z_1 + \alpha_2(1 + s(Z_2))$ in (15) negative, and dampens economic growth $\gamma = \frac{A-\rho}{\sigma}$. This may cause a hump-shaped relationship between the money supply and economic growth. When the inflation rate is low, monetary expansion has a positive effect on economic growth (a Tobin effect), but sufficiently high inflation, on the contrary, decreases the rate of economic growth (a reverse Tobin effect).

We formally present the following proposition.

Proposition 3: When the government can commit to setting $\mu = \left(\frac{\alpha_1}{\alpha_2} - 1\right)s'(Z_2)$, such a monetary policy locally maximizes the economic growth rates.

Proof: See the Appendix.

When $\alpha_1 \leq \alpha_2$, the optimal rate of money is greater than zero. As discussed in the final part of Section 3, there exists a unique BGP when $\mu \geq 0$ and $s(Z_2) = s_0(Z_2)^{-\eta}$. This specification is used in Section 6.

In particular, when $\alpha_1 = \alpha_2$ ($\xi = \alpha_1(Z_1 + Z_2)$), the economic growth rate γ is maximized by $\mu = 0$. That is, if the degree of impatience increases in the economy-wide average ratio of total assets (the sum of capital and money holdings) to consumption, then a zero rate of growth of the money supply achieves maximized endogenous growth. Notice that the inflation rate is $-\gamma < 0$ and the nominal rate of interest is $A - \gamma$ under the optimal policy.

We should notice that this proposition does not always guarantee that the government is able to print money at the rate of $\left(\frac{\alpha_1}{\alpha_2} - 1\right)s'(Z_2)$. When $\alpha_1 > \alpha_2$, the optimal growth rates of money is negative, and the rate could be smaller than $\frac{A(1-\sigma)}{\sigma}$. Recall that $\mu \geq \frac{A(1-\sigma)}{\sigma}$ is assumed for a positive nominal rate of interest. Furthermore, the proposition describes only the local properties of optimal monetary policy. In Section 6, we conduct numerical examples to demonstrate the global hump-shaped relation with a set of parameters.

We should also notice that the monetary policy maximizing the economic growth does not necessarily maximize the level of welfare. Given k_0, the indirect utility of the BGP is:

$$W = \int_0^\infty \frac{c^{1-\sigma}}{1-\sigma} e^{-\Delta_t} dt = \frac{\sigma\left(\frac{k_0}{Z_1}\right)^{1-\sigma}}{(1-\sigma)\left((\sigma-1)A+\rho\right)}.$$

We can obtain the effect on W with respect to μ :

$$\frac{dW}{d\mu} = \frac{\sigma\left(\frac{k_0}{Z_1}\right)^{1-\sigma}}{(\sigma-1)A+\rho}\left(-\frac{1}{Z_1}\frac{dZ_1}{d\mu} + \frac{\sigma}{(\sigma-1)A+\rho}\frac{d\gamma}{d\mu}\right). \tag{16}$$

The first term in the bracket represents the effect of reducing initial consumption and takes a negative value, whereas the second represents the effect of economic growth. Equation (16) is negative at the rate of μ such that $\frac{d\gamma}{d\mu} = 0$.

Naturally, we wonder under what conditions monetary expansion improves welfare. If the second term in the bracket of (16) is sufficiently large and positive, $\frac{dW}{d\mu}$ may be positive. This would be more likely when σ is closer to one and $\frac{d\gamma}{d\mu}$ is sufficiently large and positive. When $\alpha_1 = 0$, in which monetary growth always raises the BGP, then the necessary and sufficient condition for $\frac{dW}{d\mu} = 0$ is:

$$\sigma s' - \alpha_2 Z_1 \rho' + \frac{\alpha_2 \sigma (1+s)\rho'}{(\sigma-1)\left((\sigma-1)A+\rho\right)} = 0,$$

to which it is hard to attach more economic explanations. Furthermore, this condition does not always guarantee the existence of welfare-improving μ . We present a numerical welfare-improving example in Section 6.

Finally, we consider the relationship between money growth and inflation. Because $\pi = \mu - \gamma$,

$$\frac{d\pi}{d\mu} = 1 - \frac{d\gamma}{d\mu} = s' \frac{\sigma(1+s) + \alpha_1 \rho' Z_1^2}{\sigma s'(1+s) + \alpha_1 \rho'\left(s'' Z_1^2 + s' Z_1\right) + \alpha_2 \rho'(1+s)} = -s' \frac{dZ_2}{d\mu} > 0.$$

That is, inflation and money supply always have a positive correlation. When $\frac{d\gamma}{d\mu} > 0$ (resp.), monetary expansion accelerates the rate of inflation more (resp. less) than proportionately. When $\frac{d\gamma}{d\mu} = \mathbf{0}$, then $\frac{d\pi}{d\mu} = \mathbf{1}$. Although Proposition 3 shows the relationship between economic growth and the money supply rate, Section 6 examines the relationship between economic growth and inflation rates numerically.

5. Local Dynamics

In this section, we briefly mention the local stability of the balanced growth path to examine whether the path is stable in an economic sense.

Remember that $Z_1 = \frac{k}{c}, Z_2 = \frac{m}{c}$ and s is a function of Z_2 alone. Manipulating equations (8), (9), and (10) yields the following differential equation system of Z_1 and Z_2:

$$\begin{bmatrix} \frac{\dot{Z}_1}{Z_1} \\ \frac{\dot{Z}_2}{Z_2} \end{bmatrix} = \begin{bmatrix} \sigma & \tau \\ 0 & \sigma + \tau \end{bmatrix}^{-1} \begin{bmatrix} \sigma \left\{ A - \frac{\{1 + s(Z_2)\}}{Z_1} \right\} - A + \rho(\xi) \\ \sigma\{\mu + A + s'(Z_2)\} - A + \rho(\xi) \end{bmatrix}$$

where

$$\tau(Z_2) = \frac{s''(Z_2)Z_2^2}{1 + s(Z_2) - s'(Z_2)Z_2},$$

and $\xi = \alpha_1 Z_1 + \alpha_2 Z_2$, respectively.

Linearization with respect to Z_1 and Z_2 leads to:

$$\begin{bmatrix} \frac{\dot{Z}_1}{Z_1} \\ \frac{\dot{Z}_2}{Z_2} \end{bmatrix} = J \begin{bmatrix} Z_1 - Z_1^* \\ Z_2 - Z_2^* \end{bmatrix},$$

where J is the 2×2 Jacobian matrix of the dynamic system around Z_1^* and Z_2^*. With some algebra, we can show that:

$$\det J = -\frac{\Omega}{\{\sigma + \tau(Z_2)\}}$$

for each case $X = \alpha_1 K + \alpha_2 M$, where Ω is defined in the previous section for each case[8]. Because $\sigma > 0$ and $\tau(Z_2) > 0$, $\det J$ and Ω have opposite signs. Similarly, the trace takes[9]:

---

[8] The detailed derivation is available by request from the author.

$$\mathrm{tr}J = \frac{1+s}{Z_1^2} + \frac{(\alpha_1 + \alpha_2)\rho' + \sigma s''}{\sigma + \tau}.$$

Our dynamic system before reduction has two jump variables c and m. Therefore, Z_1 and Z_2 should have unstable roots for the characteristic function for economic stability. That is, the economic stability condition is $\det J > 0$ and $\mathrm{tr}J > 0$. Both hold when $\Omega < 0$ and $\rho' > 0$. As discussed in the previous section, the term Ω is negative as long as $s''(Z_2) + \frac{s'(Z_2)}{Z_1} > 0$. In Section 6, we confirm that $\Omega < 0$ with a plausible set of parameters.

6. Numerical Exercises

In this section, we conduct numerical exercises using several sets of parameters[10]. Remember that the felicity function is assumed to be $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, $\sigma > 1$ for a BGP. In addition, the TC function is specified as $s\left(\frac{m}{c}\right) = s_0\left(\frac{m}{c}\right)^{-\eta}$ for $\frac{m}{c} > 0$, where $s_0 > 0$ and $\eta > 0$.

Furthermore, for simplicity ρ is assumed to be linear $(\rho'' = 0)$. The discount rate function is represented as $\rho(\xi) = \rho_0 + \rho_1(\xi - \xi_0)$, where $\xi = \alpha_1 Z_1 + \alpha_2 Z_2$ and ξ_0 is obtained from the corresponding BGP with a benchmark rate of money growth μ_0 and a constant time preference ρ_0. Clearly $\rho'(\xi) = \rho_1 > 0$. To satisfy the assumption for Proposition 2, we assume that $\rho(\xi) = \epsilon > 0$ if $\rho_0 + \rho_1(\xi - \xi_0) < \epsilon$ and $\rho(\xi) = A - \epsilon$ if $\rho_0 + \rho_1(\xi - \xi_0) > A - \epsilon$.

We should specify all the parameters σ, ρ_0, ρ_1, μ_0, s_0, η, A, and ϵ. We calibrate these parameters so that $Z_1 = 4$[11], $Z_2 = 1$, $\gamma = 0.1$ and $\pi = 0.05$ on the BGP. The ratio of capital to consumption is 4:1, and the amounts of money and consumption are equivalent on the BGP. The economy is growing at 10 %. We set the coefficient of relative risk aversion σ at 3, which is a plausible value in terms of empirical evidence. With the endogenous growth rate of 10 %, the growth rate of money supply μ_0 should be 15 %. The remaining parameters s_0, η, and A are determined from (12), (13), and (14): $s_0 = 0.2$, $\eta = 2.25$, and $A = 0.4$[12]. Remember that $\Omega < 0$, discussed in Sec-

---

[9] The detailed derivation is available by request from the author.
[10] The Matlab codes are available by request from the author.
[11] The ratio of capital to consumption is a little higher than observed ratios ranging from 2 to 3. In an endogenous growth model with linear technology, physical capital implicitly includes human capital.
[12] The value corresponding to the real rates of interests seems higher than that observed.

tion 4, if $\rho' > 0$ and $s''(Z_2) + \frac{s'(Z_2)}{Z_1} > 0$. The last condition is equivalent to $\frac{Z_2}{Z_1} \leq \frac{Z_2 s''}{s'} = \eta + 1$, which is satisfied in our transaction function ($\frac{Z_2}{Z_1} = 0.25 < \eta + 1 = 3.25$).

When $\rho_1 = 0$ or $\rho = \rho_0$, the endogenous economic growth is five percent, independent of growth rates of monetary supply or inflation. Apart from $\rho_1 = 0$, we investigate the relationship between inflation and economic growth rates. We assume $\rho_1 = 0.1$ so that $0 < \rho < A$ for a specific wide range of μ in the benchmark case. To our best knowledge, there is no standard value for ρ_1. Later we examine the sensitivity analysis with respect to ρ_1 as well as σ . We also set ϵ at 0.01 . The benchmark values of the parameters are summarized in Table 1. With this set, we can find a nondegenerate BGP ($\gamma > 0$, $Z_1 > 0$, $Z_2 > 0$) for a sufficiently wide rage of μ .

Table 1

A Set of Parameters

σ	ρ_0	ρ_1	μ_0	s_0	η	A	ϵ
3	0.1	0.1	0.15	0.2	2.25	0.4	0.01

Figure 1 considers the case where $X = K$ ($\alpha_1 = 1$ and $\alpha_2 = 0$) on the BGP. With this benchmark set of parameters, $\Omega < 0$ as discussed in Section 4, or $\det J > 0$ as discussed in Section 5. As examined in Section 4, $\rho'\left(\frac{K}{C}\right) = 0.1 > 0$ indicates $\frac{d\gamma}{d\mu} < 0$ and $\frac{d\pi}{d\mu} > 0$. Thus, there is a negative relationship between inflation and economic growth. Figure 2 depicts the relationship between inflation and economic growth in the case where $X = M$ ($\alpha_1 = 0$ and $\alpha_2 = 1$). As discussed in Section 4, monetary expansion policy increases the rate of economic growth as well as the rate of inflation.

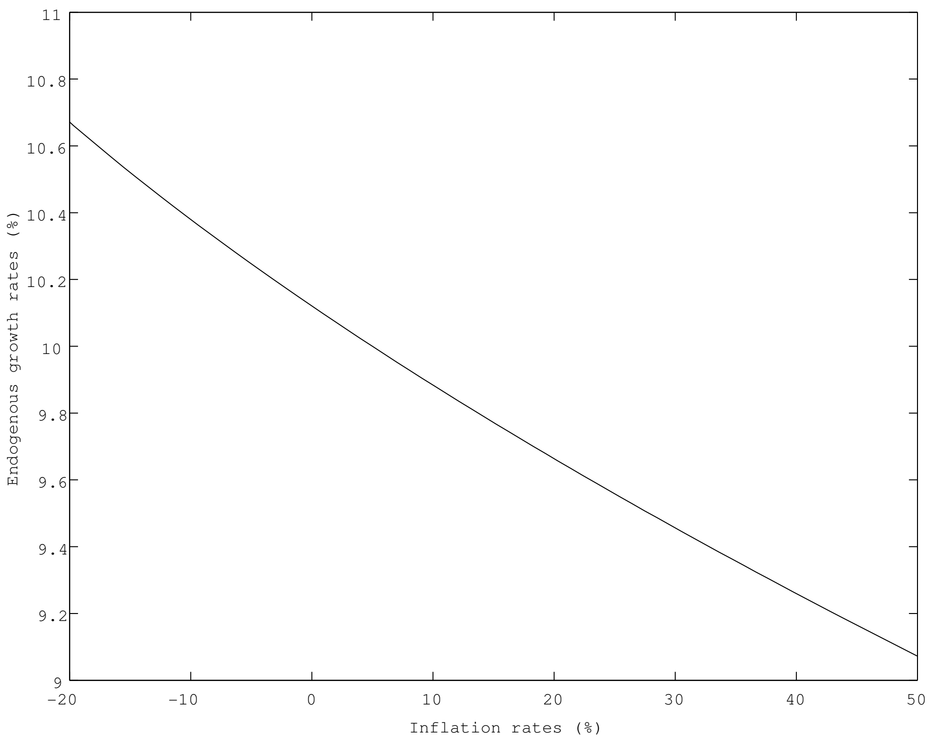
Figure 3 illustrates the relationship between inflation and economic growth in the case where $X = K + M$ ($\alpha_1 = \alpha_2 = 1$). As shown in Proposition 3, the endogenous economic growth rate is 10.07% maximized at zero growth rate of money supply. Reducing the rate of money supply from 15 % to 0% increases the endogenous growth by only 0.07 %. Because zero monetary expansion implies that the rate of inflation is equal to the negative rate of growth, Figure 3

However, as discussed in footnote 6, the parameter A indicates returns on not only observed physical capital and but also on human capital.

shows that the relationship is hump-shaped at the inflation rates of -10.07 % with the given set of parameters.

Figure 1

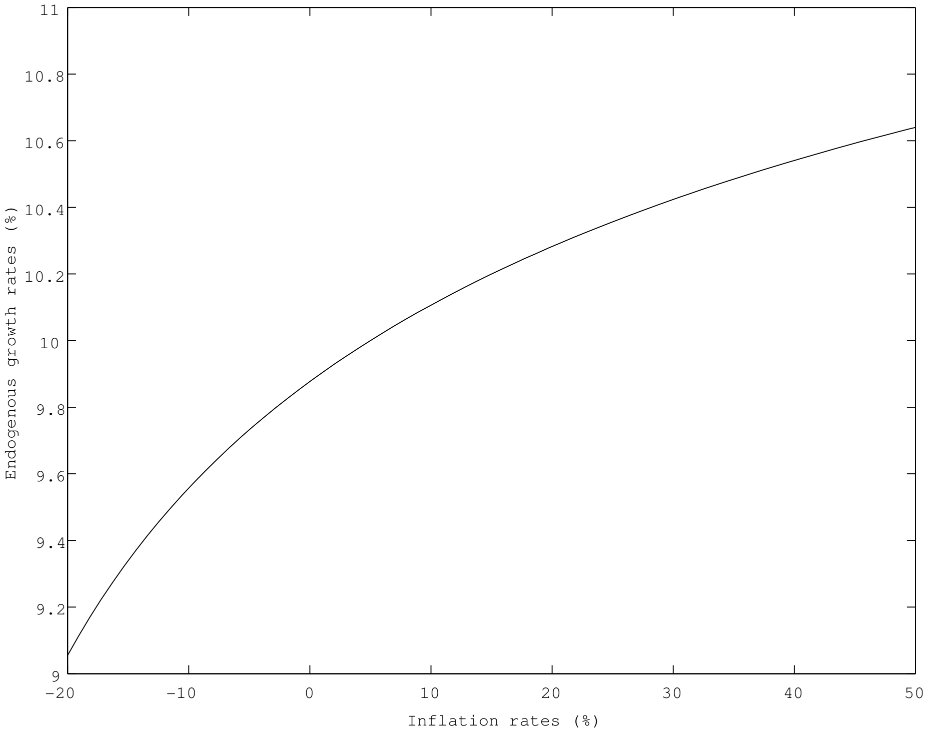
The relation on the BGP between inflation and economic growth in the case of $\alpha_1 = 1$ and $\alpha_2 = 0$



Notes: The rate of economic growth is 10% when the money supply rate is 15%.

Figure 2

The relation on the BGP between inflation and economic growth in the case of $\alpha_1 = 0$ and $\alpha_2 = 1$



Notes: The rate of economic growth is 10% when the money supply rate is 15%.

Figure 4 illustrates the relationship between inflation and economic growth in the case where $X = AK + M$ ($\alpha_1 = 0.4$ and $\alpha_2 = 1$). This case implies that the agent is affected by the sum of real balances and income. As presented in Proposition 3, endogenous growth is maximized when the rate of money supply is $\mu = 0.6 s_0 \eta Z_2^{-\eta-1} = 0.45$. The maximized economic growth rate is 10.09 % The corresponding rate of inflation is 35.9 %. This numerical example indicates that a certain level of positive inflation can achieve optimal economic growth.

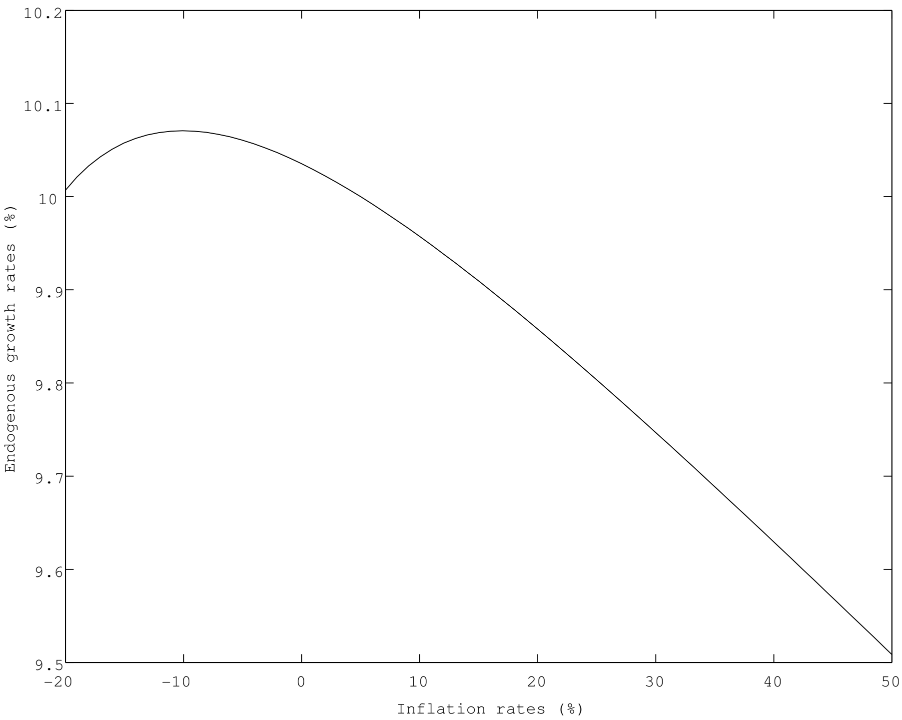
We examine the robustness of the parameter regarding the marginal effect of the relative assets on the degree of impatience, ρ_1. Figure 5 compares the case of $\rho_1 = 0.1$ with that of $\rho_1 = 0.2$ [13]. Both cases consider $X = K + M$

[13] When ρ_1 takes 0.3 or more, the discounting rate ρ applies to the lowest value at zero growth rate of money supply.

($\alpha_1 = 1$ and $\alpha_2 = 1$), and both maximize economic growth through zero money growth. The rate of endogenous growth is 10.07 % in the case of $\rho_1 = 0.1$, whereas the rate is 10.11 % in the case of $\rho_1 = 0.2$ [14]. The steeper the slope of discounting rate, the stronger is the relationship between inflation and economic growth. However, even with doubled ρ_1, the monetary policy only improves the rate of economic growth by less than 0.4 %. Similarly, Figure 6 compares the cases of $\sigma = 3$ and of $\sigma = 5$. A larger degree of relative risk averseness or smaller elasticity of intertemporal substitution makes the effect of monetary policy on economic growth weaker.

Figure 3

The relation on the BGP between inflation and economic growth in the case of $\alpha_1 = 1$ and $\alpha_2 = 1$

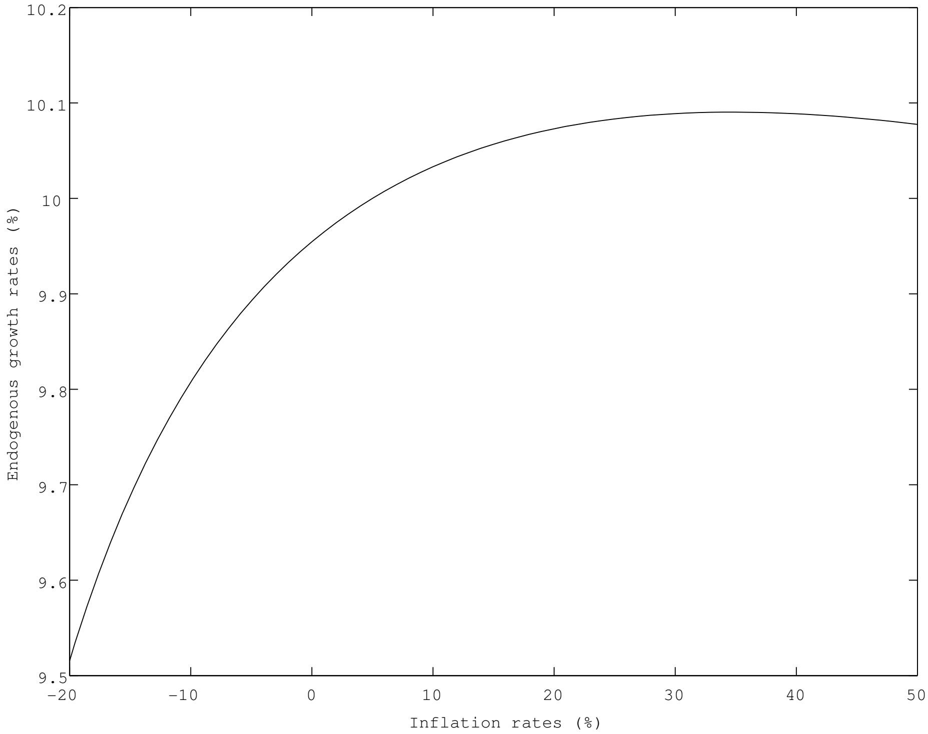


Notes: The rate of economic growth is 10% when the money supply rate is 15%.

[14] Notice that this case satisfies $Z_1 = 4$ and $Z_2 = 1$ at the rate of $\mu_0 = 0.15$, and the other parameters are unchanged.

Figure 4

The relation on the BGP between inflation and economic growth in the case of $\alpha_1 = 0.3$ and $\alpha_2 = 1$



Notes: The rate of economic growth is 10% when the money supply rate is 15%.

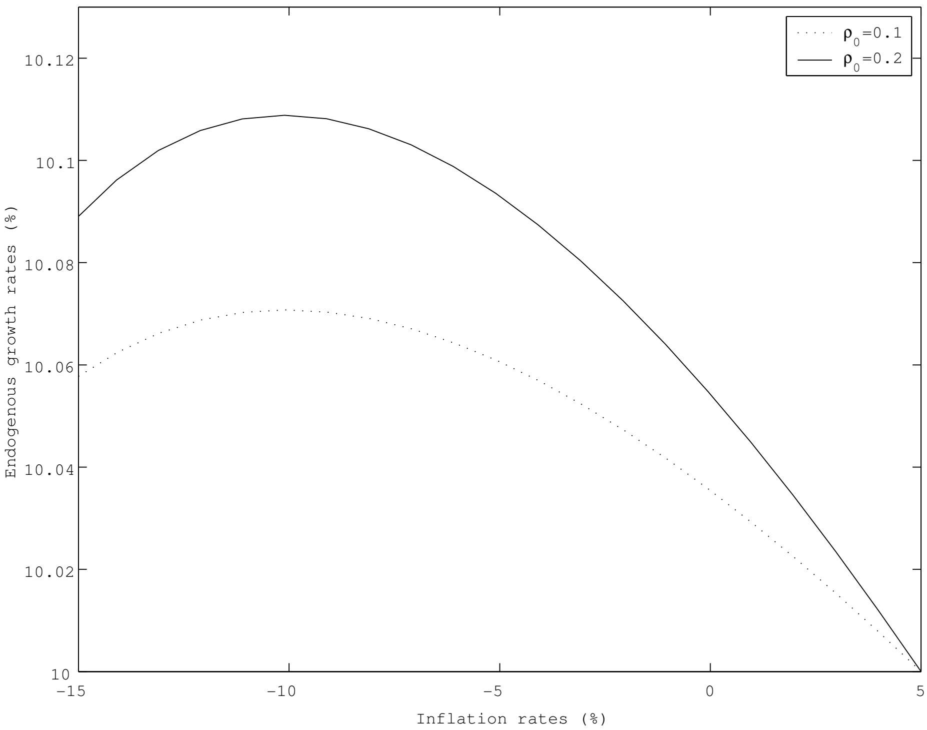
As discussed in Section 4, the monetary policy maximizing economic growth does not necessarily maximize the level of welfare. As conjectured, when σ is close to one, a welfare-improving monetary policy is more likely to exist. To demonstrate this in a numerical way, consider $X = M$ ($\alpha_1 = 0$ and $\alpha_2 = 1$), in which inflation and economic growth are positively correlated. Figure 7 compares the case of $\sigma = 3$ with that of $\sigma = 1.8$ in terms of welfare[15]. Notice that unlike the above two robustness examinations, this exercise uses the other parameters, except for $\sigma = 1.8$ from Table 1, and we cannot obtain a set of parameters satisfying $Z_1 = 4$ and $Z_2 = 1$ at the rate of $\mu_0 = 0.15$. Whereas inflation always lowers welfare in the case of $\sigma = 3$, inflation and welfare have a

[15] To calculate the level of welfare, we need to set the value of initial capital k_0, and choose $k_0 = 1$.

hump-shaped relationship in the case of $\sigma = 1.8$. Welfare is maximized when the rate of monetary growth is 35% , the economic growth rate is 31.2% and the inflation rate is 3.8% .

Figure 5

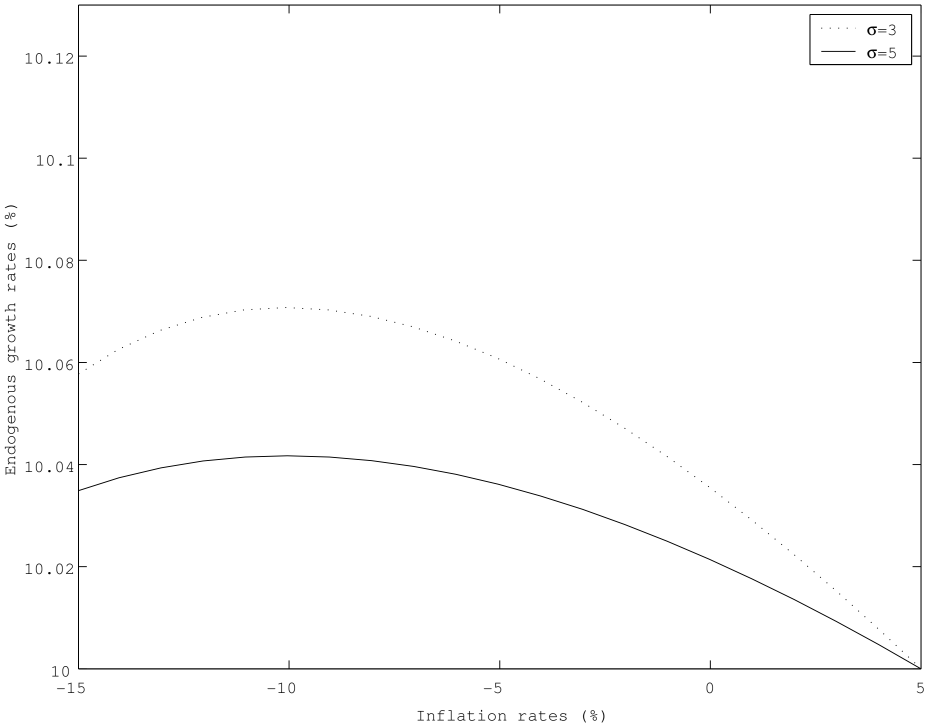
The economic-growth comparison on the BGP between the cases with $\rho_1 = 0.1$ and $\rho_0 = 0.2$



Notes: The rate of economic growth is 10% when the money supply rate is 15%.

Figure 6

The economic-growth comparison on the BGP between the cases with $\sigma = 3$ and $\sigma = 5$

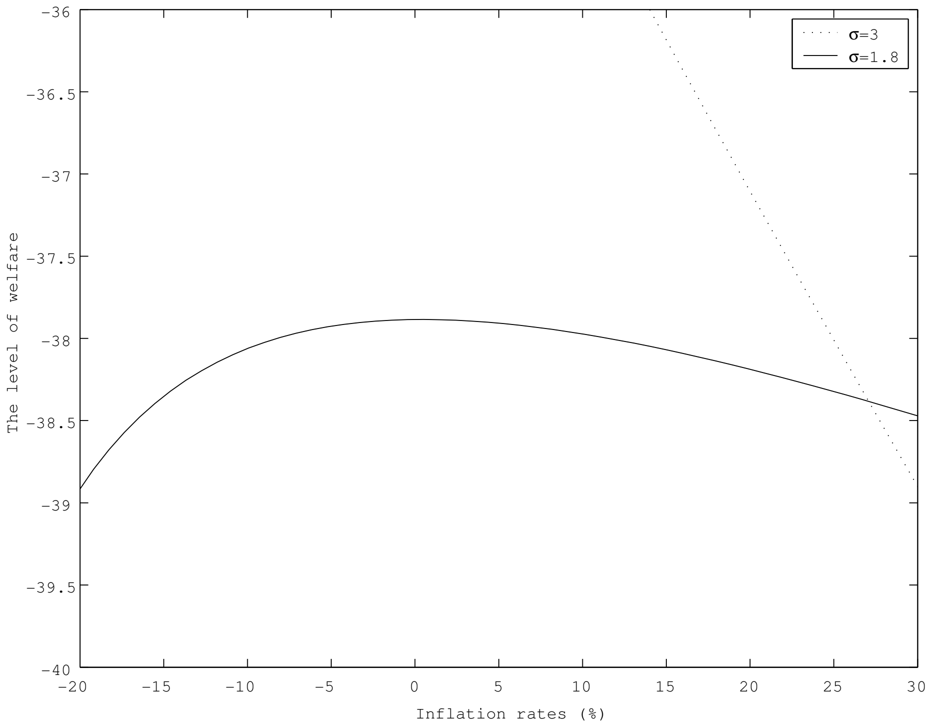


Notes: The rate of economic growth is 10% when the money supply rate is 15%.

In sum, we have presented the relationship between inflation and economic growth using the TC model with varying discount rates. It should be noted that the effect of monetary policy is not very large with the benchmark set of parameters. Figures 1 and 2 show that an increase of approximately 50 % in inflation raises or lowers economic growth by around 1%, and Figures 3 and 4 demonstrate that an increase of approximately 20 % in inflation has less than 0.2 % impact on economic growth. Money is not superneutral, but the magnitude is not very large in our specification of parameters and functional forms.

Figure 7

The welfare comparison on the BGP between the cases with $\sigma = 3.0$ and $\sigma = 1.8$



7. Comparison with the Model with Internally Determined Discount Rates

This section compares the previous model with one in which consumption at the discount rates is determined internally by an economic agent. The model environment follows the model we used except for the discount rate being characterized by

$$\dot{\Delta} = \rho\left(\frac{X}{c}\right), \tag{17}$$

instead of (2), where $X = \alpha_1 K + \alpha_2 M$. The discount rate is determined by the ratio of aggregate assets to individual consumption. We should note that the above discount rate is constant on the BGP.

Given X , the economic agent chooses c , k , and m to maximize (1) subject to (3) and (17). To solve the problem, we consider the Hamiltonian:

$$\frac{c^{1-\sigma}}{1-\sigma}) \tag{18}$$

$$+\lambda(Ak-\pi m+v-[1+s(m/c)]c)-\phi\rho(X/c)+\psi(a-m-k)\}$$

where λ and ϕ are the costate variables of (3) and (17), respectively. Pontryagin's maximum principle yields (5), (6), and

$$c^{-\sigma}-\left\{1+s\left(\frac{m}{c}\right)-\frac{s'\left(\frac{m}{c}\right)m}{c}\right\}\lambda+\frac{\phi\rho' X}{c^2}=0, \tag{19}$$

$$\dot{\phi}=-u+\phi\rho.$$

The initial conditions are $k_0>0$, $\tilde{m}_0>0$ and $P_0=1$, and the transversality condition of Michel (1982) is $\lim_{t\to\infty}H(t)=0$.

As discussed in Palivos et al. (1997), the Hamiltonian is independent of time on the optimal path, and the above transversality condition implies that the value of the Hamiltonian is zero on the optimal path. Hence,

$$\phi=\frac{u(c)+\lambda\left(Ak-\pi m+v-\left[1+s\left(\frac{m}{c}\right)\right]c\right)+\psi(a-m-k)}{\rho\left(\frac{X}{c}\right)}.$$

This is interpreted as the lifetime utility on the optimal path. Using (19), it is easy to show $\phi_c=0$ or equivalently

$$\lambda=c^{-\sigma}\frac{1+\frac{\frac{\rho' X}{c}}{\{\rho(1-\sigma)\}}}{1+s\left(\frac{m}{c}\right)-\frac{s'\left(\frac{m}{c}\right)m}{c}-\left(\frac{\rho'}{\rho}\right)\left(\frac{\dot{a}X}{c^2}\right)}.$$

The dynamic system of c , m , and k under a monetary equilibrium is characterized by (6), (8), (10), and

$$\lambda=c^{-\sigma}\frac{1+\frac{\rho'\xi}{\{\rho(1-\sigma)\}}}{1+s(Z_2)-s'(Z_2)Z_2-\left(\frac{\rho'}{\rho}\right)(Z_1+Z_2)\xi\left(\frac{\dot{a}}{a}\right)}, \tag{20}$$

with the boundary conditions, where $Z_1=\frac{k}{c}$, $Z_2=\frac{m}{c}$, and $\xi=\alpha_1 Z_1+\alpha_2 Z_2$.

The dynamic system of the model with externally determined discount rates is characterized by (8), (9), and (10). The only difference from the model with internally determined discount rates is the shadow price, λ . In the case of externally determined discount rates, combining (6) and the time difference with respect to λ yields (9). In the case of internally determined discount rates, it is difficult to describe the time difference with respect to (20) with $\xi = \alpha_1 Z_1 + \alpha_2 Z_2$, and therefore it is difficult to characterize the dynamic system of capital and consumption.

On the BGP, however, equation (20) indicates that $\frac{\dot{\lambda}}{\lambda} = -\frac{\sigma \dot{c}}{c} = -\sigma\gamma$. Thus, the dynamic system of equations (6), (8), (10), and (20) are reduced to (12), (13), and (14) on the BGP. Thus, we can obtain the following proposition:

Proposition 4: Consider the model with the discount rate function determined by (17). Then, γ, Z_1, and Z_2 on the balanced-growth path are exactly the same as those of the model with discount rates socially determined by $\rho\left(\frac{X}{C}\right)$ where $X = \alpha_1 K + \alpha_2 M$.

Results of the comparative statics of the BGP in the two models with externally and internally determined discount rates are equivalent.

Many studies dealing with internally determined discount rates assume that the degree of impatience increases with consumption[16]. This assumption is less plausible from an empirical viewpoint, but it is imposed for dynamic stability. On the other hand, the previous model assumes increasing impatience in the ratio of aggregate generalized assets to consumption to produce the hump-shaped relationship between the money supply and economic growth. Replacing economy-wide consumption with individual consumption implies that the degree of impatience is decreasing in individual consumption given average generalized assets, which supports empirical evidence, including that of Becker and Mulligan (1997).

However, we have two caveats to this proposition. First, the equivalent result holds only on the BGP. When we attempt to conduct a dynamic analysis, we must differentiate λ in (20) with respect to time. Even in a local stability analysis, the dynamic system is too complicated, and so we would have to resort to numerical investigation.

Next, the equivalent result for the proposition does not hold even on the BGP when an individual variable other than consumption determines the discount rate. For example, consider

$$\dot{\mathbf{\Delta}} = \rho\left(\frac{m}{c}\right).$$

The degree of impatience is determined by relative individual real balances. Then Pontryagin's maximum principle yields (4), (6), and

$$\lambda(A + \pi + s') + \frac{\phi\rho'}{c} = 0$$

[16] Das (2003) investigated the case of decreasing impatience with consumption, but her interest lay only in the exogenous growth model.

instead of (5). With some algebra[17], we can show that the BGP characterizes (12), (14), and

$$\left\{1-\frac{\rho'}{\rho}\frac{(1-\sigma)(Z_1+Z_2)}{(1-\sigma)+\frac{Z_2\rho'}{\rho}}\right\}\gamma = A+\mu+s'+\frac{\rho'}{\rho}\left\{\frac{1+s(Z_2)-s'(Z_2)Z_2}{(1-\sigma)+\frac{Z_2\rho'}{\rho}}\right\}$$

instead of (13). Thus, it is much more difficult to find conditions for the existence of BGP.

8. Concluding Remarks

To explore the relationship between inflation and economic growth, we have used the TC model with a socially determined discount rate and linear production technology. We have defined a BGP and proved that there uniquely exists a nondegenerate BGP when we assume increasing impatience in the ratio of general assets to consumption and several other conditions, and that such a BGP is locally stable. We have found that inflation affects the endogenous growth in the case of nonconstant time preferences. Specifically, if the degree of impatience increases in the economy-wide average total assets to consumption ratio, then a zero rate of growth of the money supply achieves maximized endogenous growth.

We have conducted several numerical exercises to confirm the hump-shaped relationship between the rate of inflation and the rate of economic growth with a set of plausible parameters. In particular, we have demonstrated such a hump-shaped relationship, but we have discovered that the impact of inflation on economic growth is quantitatively small. Finally, we have compared our model to one with discount rates determined internally by the individual, and we have proved that if the discount rate depends on the individual consumption, the results of the comparative statics on the BGP are observationally equivalent in both models.

We point out two extensions. First, whereas we have used the TC model in this study, we would apply our theory to the MIUF model. Feenstra (1986) demonstrated functional equivalence between the TC and MIUF models with an inelastic labor supply and a constant time preference in an exogenous growth framework. We need to examine whether such a functional equivalence is established in our framework. When an equivalent functional form in the MIUF is not appropriate, quantitative equivalence (Wang and Yip, 1992) is worth investigating. When we find a proper functional form in which a nondegenerate monetary BGP exists, then it would be interesting to examine whether inflation and economic growth still have a hump-shaped relationship.

[17] The detailed derivation is available by request from the author.

Second, although we have concluded that the impact of inflation on economic growth is small, our choice of parameters and functional forms does not capture the real economy very well. In particular, we should estimate the marginal effect of the relative assets on the degree of impatience, ρ_1, in a more accurate way. Many empirical studies including Becker and Mulligan (1997) used microeconomic data to estimate the marginal effect of assets on time preferences. For the calibration of our model, international macroeconomic panel data would be more suitable. Empirical analysis using such data must be another future task.

Appendix

Proof of Proposition 2:

From (12), $\gamma = \frac{A-\rho(\xi)}{\sigma} > \frac{A-\bar{\rho}}{\sigma} > 0$. We combine (12), (13), and (14) to reduce two equations:

$$\rho(\alpha_1 Z_1 + \alpha_2 Z_2) = (1-\sigma)A - \sigma\mu - \sigma s'(Z_2), \tag{21}$$

$$-\frac{1+s(Z_2)}{Z_1} = \mu + s'(Z_2). \tag{22}$$

It suffices for the proof to examine whether there exists a pair of $Z_1 > 0$ and $Z_2 > 0$ satisfying (21) and (22).

Set an arbitrary $\mu \geq \frac{(1-\sigma)A}{\sigma}$ to be fixed. Then, by the property of s', there exists a $\bar{Z}_2 > 0$ such that $s'(\bar{Z}_2) + \mu - \frac{A(1-\sigma)}{\sigma} = 0$. Note that $\bar{Z}_2$ can take a value of infinity when $\mu = \frac{(1-\sigma)A}{\sigma}$.

Because $s'(Z_2) + \mu < s'(\bar{Z}_2) + \mu = \frac{A(1-\sigma)}{\sigma} < 0$ for all $Z_2 \in (0, \bar{Z}_2)$, equation (22) is rewritten as:

$$Z_1 \equiv \zeta(Z_2) = -\frac{1+s(Z_2)}{\mu + s'(Z_2)}, \tag{23}$$

with $\lim_{Z_2 \to 0} \zeta(Z_2) = 0$ and

$$\lim_{Z_2 \to \bar{Z}_2} \zeta(Z_2) = -\frac{1+s(\bar{Z}_2)}{\left(\mu + s'(\bar{Z}_2)\right)} = \frac{\sigma\{1+s(\bar{Z}_2)\}}{(\sigma-1)A} > 0.$$

The derivative $\zeta'(Z_2)$ is:

$$\zeta'(Z_2) = \frac{s''(1+s) - s'(\mu + s')}{(\mu + s')^2}.$$

That is, $\zeta'(Z_2) > 0$ or $\zeta(Z_2)$ increases monotonically if $s''(1+s) - s'(\mu + s') > 0$ for all Z_2.

To examine equation (21), we consider the case with nonconstant ρ. Substituting (23) into (21) leads to:

$$\rho(\alpha_1 \zeta(Z_2) + \alpha_2 Z_2) = (1-\sigma)A - \sigma\mu - \sigma s'(Z_2). \qquad (24)$$

Then, the right-hand side of (24) decreases in $Z_2 \in (0, \bar{Z}_2)$ with:

$$\lim_{Z_2 \to 0} (1-\sigma)A - \sigma\mu - \sigma s'(Z_2) \to \infty,$$

$$\lim_{Z_2 \to \bar{Z}_2} (1-\sigma)A - \sigma\mu - \sigma s'(Z_2) \to 0.$$

Thus, as long as $0 < \rho < A$, there exists at least one Z_2 satisfying (24).

When ρ is constant, or an increasing function only of Z_2 ($\alpha_1 = 0$), then we can obtain a unique $Z_2 \in (0, \bar{Z}_2)$. From (23), we can obtain a unique $Z_1 > 0$.

Next, consider the case where $\alpha_1 > 0$ and $\rho'(\xi) \geq 0$. When $s''(1+s) - s'(\mu + s') > 0$, then $\zeta(Z_2)$ increases monotonically. Therefore, the left-hand side of (24) increases in $Z_2 \in (0, \bar{Z}_2)$ with:

$$\lim_{Z_2 \to 0} \rho(\alpha_1 \zeta(Z_2) + \alpha_2 Z_2) \to \rho(0) > 0,$$

$$\lim_{Z_2 \to \bar{Z}_2} \rho(\alpha_1 \zeta(Z_2) + \alpha_2 Z_2) \to \rho\left(\frac{\alpha_1 \sigma\{1 + s(\bar{Z}_2)\}}{\{(\sigma - 1)A\}} + \alpha_2 \bar{Z}_2\right) > \rho(0).$$

Thus, there exists a unique Z_2 satisfying (24), giving a unique $Z_1 > 0$ from (23).

Proof of Proposition 3:

As discussed above, the first derivative of γ with respect to μ is zero when $\alpha_2(1 + s(Z_2)) + \alpha_1 s'(Z_2) Z_1 = 0$. Because $\mu + s'(Z_2) + \frac{1 + s(Z_2)}{Z_1} = 0$ from (13) and (14), $\frac{d\gamma}{d\mu} = 0$ at $\mu = \left(\frac{\alpha_1}{\alpha_2} - 1\right) s'(Z_2)$. Because $\frac{dZ_1}{d\mu} = \frac{\frac{\sigma s'}{Z_1} - \alpha_2 \rho'}{\Omega}$

and $\frac{dZ_2}{d\mu} = \frac{\alpha_1\rho' + \frac{\sigma(1+s)}{Z_1^2}}{\Omega}$, the second derivative of γ with respect to μ at $\mu = \left(\frac{\alpha_1}{\alpha_2} - 1\right)s'(Z_2)$ is:

$$\frac{d^2\gamma}{d\mu^2} = -\frac{\rho'}{Z_1^2\Omega}\frac{d\left[\alpha_2\left(1+s(Z_2)\right) + \alpha_1 s'(Z_2)Z_1\right]}{d\mu} - \frac{\left[\alpha_2\left(1+s(Z_2)\right) + \alpha_1 s'(Z_2)Z_1\right]d}{d\mu}\frac{\rho'}{Z_1^2\Omega}$$

$$= -\frac{\rho'}{Z_1^2\Omega}\left[\alpha_1 s'\frac{dZ_1}{d\mu} + \frac{\{\alpha_2 s' + \alpha_1 s'' Z_1\}(dZ_2)}{d\mu}\right]$$

$$= -\frac{\rho'}{Z_1^2\Omega^2}\left[\alpha_1 s'' Z_1\left(\alpha_1\rho' + \frac{\sigma(1+s)}{Z_1^2}\right)\right] = -\frac{\rho'}{Z_1^2\Omega}\alpha_1 s'' Z_1\frac{dZ_2}{d\mu}.$$

Because $\Omega < 0$, $\rho' > 0$, and $\frac{dZ_2}{d\mu} < 0$, then $\frac{d^2\gamma}{d\mu^2}$ is negative; γ is locally maximized in the neighborhood of $\mu = \left(\frac{\rho_1}{\rho_2} - 1\right)s'(Z_2)$.

Bibliography

1. Ahmed, S., and J. H. Rogers (2000) «Inflation and the great ratios: Long term evidence from the US» *Journal of Monetary Economics*, 45: 3–35.
2. Becker, G. S., and C. B. Mulligan (1997) «The endogenous determination of time preference» *Quarterly Journal of Economics*, 122: 729–758.
3. Benhabib, J., and R. Farmer (1994) «Indeterminacy and increasing returns» *Journal of Economic Theory*, 63: 19–41.
4. Bullard, J., and J. Keating (1995) «The long-run relationship between inflation and output in postwar economies» *Journal of Monetary Economics*, 51: 477–496.
5. Chen, B.-L., M. Hsu, and C.-H. Lin (2008) «Inflation and growth: impatience and a qualitative equivalent» *Journal of Money, Credit, and Banking*, 40: 1310–1323.
6. Das, M. (2003) «Optimal growth with decreasing marginal impatience» *Journal of Economic Dynamics and Control*, 27: 1881–1898.
7. De Gregorio, J. (1993) «Inflation, taxation, and long-run growth» *Journal of Monetary Economics*, 31: 271–298.

8. Dutta, J., and Kapur, S. (1998) «Liquidity preference and financial intermediation» *Review of Economic Studies*, 65: 551–572.

9. Feenstra, R. C. (1986) «Functional equivalence between liquidity costs and the utility of money» *Journal of Monetary Economics*, 65: 271–291.

10. Fukuda, S. (1996) «Money and economic growth with increasing returns-to-scale» The Institute of Economic Research, Hitotsubashi University. Discussion Paper no. 320.

11. Gomme, P. (1993) «Money and economic growth revised: measuring the costs of inflation in an endogenous growth model.» *Journal of Monetary Economics*, 32: 51–77.

12. Itaya, J., and K. Mino (2003) «Inflation, transaction costs and indeterminacy in monetary economies with endogenous growth» *Economica*, 70: 451–470.

13. Jha, S. K., P. Wang, and C. K. Yip (2002) «Dynamics in a transactions-based monetary growth model» *Journal of Economic Dynamics and Control*, 26: 616–635.

14. Jones, L. E. and Manuellir, R. E. (1995) «Growth and the effects of inflation» *Journal of Economic Dynamics and Control*, 19: 1405–1428.

15. Kam, E. (2005) «A note on time preference and the Tobin effect» *Economics Letters*, 89: 127–132.

16. Meng, Q. (2006) «Impatience and equilibrium indeterminacy» *Journal of Economic Dynamics and Control*, 30: 2671–2692.

17. Michel, P. (1982) «On the transversality condition in infinite horizon optimal problems» *Econometrica*, 50: 975–985.

18. Palivos, T., P. Wang, and J. Zhang (1997) «On the existence of the balanced growth equilibrium» *International Economic Review*, 38: 205–224.

19. Sidrauski, M. (1967) «Inflation and economic growth», *The Journal of Political Economy*, 75: 796–810.

20. Tobin, J. (1965) «Money and economic growth» *Econometrica*, Vol. 33, pp. 671–684.

21. Wang, P., and C. K. Yip (1992) «Alternative approaches to money and growth» *Journal of Money, Credit and Banking*, 24: 553–562.