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GENERALIZATION OF CONTINUED FRACTIONS. II

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This work represents a continuation of the study of recursive fractions begun in *Mat. Metody Fiz.-Mekh. Polya,* **54**, No. 1, 57–64 (2011). We have constructed algorithms for calculating the value of the expression $P_k Q_n - P_n Q_k$, where $\frac{P_k}{Q_k}$ and $\frac{P_n}{Q_n}$ are the kth and nth rational truncations, respectively, of a certain recursive fraction. Using the values of this expression, we make some conclusions on the character and rate of the convergence of rational truncations of a recursive fraction to its value.

5. Some General Theorems on Recursive Fractions of the nth Order

In [3], one can find the proof of Theorem 2 on the relation between the determinant of the matrix

$$A = \left\| \begin{matrix} a_{11} & -a_1 & 0 & \ldots & 0 & 0 & 0 \\ a_{21} & a_{22} & -a_2 & \ldots & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n-2,1} & a_{n-2,2} & a_{n-2,3} & \ldots & a_{n-2,n-2} & a_{n-2} & 0 \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \ldots & a_{n-1,n-2} & a_{n-1,n-1} & -a_{n-1} \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{n,n-2} & a_{n,n-1} & a_{nn} \end{matrix} \right\|$$

and the parapermanent of the triangular matrix

$$B = \left\| \begin{matrix} a_{11} & & & & \\ \frac{a_1 a_{21}}{a_{22}} & a_{22} & & & \\ \ldots & \ldots & \ddots & & \\ \frac{a_1 a_{n-1,1}}{a_{n-1,2}} & \frac{a_2 a_{n-1,2}}{a_{n-1,3}} & \ldots & a_{n-1,n-1} & \\ \frac{a_1 a_{n1}}{a_{n2}} & \frac{a_2 a_{n2}}{a_{n3}} & \ldots & \frac{a_{n-1} a_{n,n-1}}{a_{nn}} & a_{nn} \end{matrix} \right\|,$$

according to which the identity

$$\det(A) = \operatorname{pper}(B) \tag{15}$$

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is true.

Remark 1. In this theorem, the parapermanent of the matrix B is defined even if there appears an indeterminacy of the type $\frac{0}{0}$ because, in its calculation, zeros are canceled, and the indeterminacy disappears.

Theorem 5. *Let* $A_{k,m}$ *be the minor of the matrix*

$$A = \begin{Vmatrix} a_{11} & -1 & 0 & \dots & 0 & 0 \\ a_{22} & a_{12} & -1 & \dots & 0 & 0 \\ a_{33} & a_{23} & a_{13} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m-1,m-1} & a_{m-2,m-1} & a_{m-3,m-1} & \dots & a_{1,m-1} & -1 \\ a_{mm} & a_{m-1,m} & a_{m-2,m} & \dots & a_{2,m} & a_{1,m} \end{Vmatrix},$$

formed after removing from it the first row and k *th column. Then the recursive relation*

$$A_{km} = -a_{1,k-1}A_{k-1,m} + a_{2,k-1}A_{k-2,m} - \ldots + (-1)^{k-2} a_{k-2,k-1}A_{2m} + (-1)^{k-1} a_{k-1,k-1}A_{1m}, \qquad (16)$$

where $3 \le k \le m$, *is true.*

This theorem, as to its essence, coincides with Theorem 4 in [3].

Remark 2. The statement of Theorem 4 in [3] can be extended to the case of matrices A where all elements lying lower than the $(n-1)$th subdiagonal, $n = 2, 3, \ldots, m-1$, are equal to zero.

Theorem 6. *Let*

$$P_m = \begin{bmatrix} a_{11} & & & & \\ \frac{a_{22}}{a_{12}} & a_{12} & & & \\ \frac{a_{33}}{a_{23}} & \frac{a_{23}}{a_{13}} & a_{13} & & \\ \dots & \dots & \dots & \ddots & \\ \frac{a_{m,m}}{a_{m-1,m}} & \frac{a_{m-1,m}}{a_{m-2,m}} & \frac{a_{m-2,m}}{a_{m-3,m}} & \dots & a_{1,m} \end{bmatrix}_m,$$

$$Q_m = \begin{bmatrix} a_{12} & & & \\ \frac{a_{23}}{a_{13}} & a_{13} & & \\ \dots & \dots & \ddots & \\ \frac{a_{m-1,m}}{a_{m-2,m}} & \frac{a_{m-2,m}}{a_{m-3,m}} & \dots & a_{1m} \end{bmatrix}_{m-1}.$$

Then the following equality is true:

$$P_r Q_s - Q_r P_s = (-1)^{s-1} A_{s+1,r}, \qquad 1 \le s < r, \tag{17}$$

where $A_{s,r}$ *is the determinant defined in the previous theorem.*

Proof. We prove equality (17) for $s = 1$ and, for this purpose, expand the determinant of the matrix A by elements of the first row:

$$\det(A) = a_{11} \begin{vmatrix} a_{12} & -1 & \dots & 0 \\ a_{23} & a_{13} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ a_{r-1,r} & a_{r-2,r} & \dots & a_{1,r} \end{vmatrix} + \begin{vmatrix} a_{22} & -1 & \dots & 0 \\ a_{33} & a_{13} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ a_{r,r} & a_{r-2,r} & \dots & a_{1,r} \end{vmatrix}.$$

The determinant of left-hand side and the first determinant of right-hand side according to Theorem 2 in [3] are equal to P_r and Q_r, respectively, and the second determinant of right-hand side is equal to A_{2r}. Hence, we have $P_r = a_{11} Q_r + A_{2r}$, where one may assume that $a_{11} = P_0$ and $1 = Q_0$.

Let equality (17) be correct for all $s < t - 1 < r$. Then, expanding the paradeterminants Q_t and P_t in the expression $P_r Q_t - Q_r P_t$ by elements of the last row and grouping the corresponding terms, we obtain

$$a_{1,t}(P_r Q_{t-1} - Q_r P_{t-1}) + a_{2,t}(P_r Q_{t-2} - Q_r P_{t-2}) + \dots + a_{t-1,t}(P_r Q_1 - Q_r P_1) - a_{t,t} Q_r$$

$$= a_{1,t}(-1)^{t-2} A_{t,r} + a_{2,t}(-1)^{t-3} A_{t-1,r} + \dots + a_{t-1,t} A_{2r} - a_{t,t} A_{1r} = (-1)^{t-1} A_{t+1,r}.$$

The following proposition follows immediately from Theorem 6 with regard for Remarks 1 and 2:

Theorem 7. *For the* r*th and* s*th rational truncations of a recursive fraction of the* n*th order,* $1 \le s < r$*, the equality*

$$\frac{P_r}{Q_r} - \frac{P_s}{Q_s} = \frac{(-1)^{s-1} A_{s+1,r}}{Q_r Q_s} \tag{18}$$

is true.

Corollary 1. *For the* r*th,* $(r-1)$*th, and* $(r-2)$*th rational truncations of recursive fractions of the* n*th order, the equalities*

$$\frac{P_r}{Q_r} - \frac{P_{r-1}}{Q_{r-1}} = \frac{(-1)^{r-2} A_{r,r}}{Q_r Q_{r-1}},$$

$$\frac{P_r}{Q_r} - \frac{P_{r-2}}{Q_{r-2}} = \frac{(-1)^{r-1} A_{r-1,r}}{Q_r Q_{r-2}}$$

are true. Here,

$$A_{r,r} = \begin{Vmatrix} a_{22} & a_{12} & -1 & \dots & 0 & 0 & \dots & 0 \\ a_{33} & a_{23} & a_{13} & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n,n} & a_{n-1,n} & a_{n-2,n} & \dots & \dots & \dots & \dots & 0 \\ 0 & a_{n,n+1} & a_{n-1,n+1} & \dots & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{n,r-1} & a_{n-1,r-1} & \dots & a_{1,r-1} \\ 0 & 0 & 0 & \dots & 0 & a_{n,r} & \dots & a_{2,r} \end{Vmatrix},$$

$$A_{r-1,r} = \begin{Vmatrix} a_{22} & a_{12} & -1 & \dots & 0 & 0 & \dots & 0 & 0 \\ a_{33} & a_{23} & a_{13} & \dots & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n,n} & a_{n-1,n} & a_{n-2,n} & \dots & \dots & \dots & \dots & 0 & 0 \\ 0 & a_{n,n+1} & a_{n-1,n+1} & \dots & \dots & \dots & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{n-2,r-3} & a_{n-3,r-3} & \dots & -1 & 0 \\ 0 & 0 & 0 & \dots & a_{n-1,r-2} & a_{n-2,r-2} & \dots & a_{1,r-2} & 0 \\ 0 & 0 & 0 & \dots & a_{n,r-1} & a_{n-1,r-1} & \dots & a_{2,r-1} & -1 \\ 0 & 0 & 0 & \dots & 0 & a_{n,r} & \dots & a_{3,r} & a_{1,r} \end{Vmatrix}.$$

The statement of this corollary follows directly from equality (18) of Theorem 7 for $s = r-1$ and $s = r-2$. Studying the constancy of signs of the matrices $A_{r,r}$ and $A_{r-1,r}$ for an arbitrary natural r, $r > 1$, is important in the investigations of two-sided estimates of rational truncations.

If $n = 2$, the second–order recursive fractions coincide with usual continued fractions, and the determinants of the matrices $A_{r,r}$ and $A_{r-1,r}$ are equal to 1 and a_{1r}, respectively, which is in good agreement with Corollary 2 of Proposition 4.2 [1, p. 86]).

6. The Best Approximations with the Help of Third-Order Recursive Fractions

For $n = 3$ and $a_{1i} = q_i$, $a_{2i} = p_i$, $a_{3i} = 1$, we write the recursive fraction (2) as a usual third-order recursive fraction

$$\left[\begin{array}{c|ccccccc}
q_1 & & & & & & & \\
\dfrac{p_2}{q_2} & q_2 & & & & & & \\
\dfrac{1}{p_3} & \dfrac{p_3}{q_3} & q_3 & & & & & \\
0 & \dfrac{1}{p_4} & \dfrac{p_4}{q_4} & q_4 & & & & \\
\dots & \dots & \dots & \dots & \ddots & & & \\
0 & 0 & 0 & 0 & \dots & q_r & & \\
\dots & \dots & \dots & \dots & \dots & \dots & \ddots &
\end{array}\right]_{\infty}, \tag{19}$$

for which the matrices $A_{r,r}$ and $A_{r-1,r}$ are Hessenberg matrices [5]. Hence, according to Theorem 2 (see [3]) on the relation of the determinant of a Hessenberg matrix with the paradeterminant of a triangular matrix, the matrices $A_{r,r}$ and $A_{r-1,r}$ can be represented as paradeterminants

$$B_r = \left\langle \begin{matrix}
p_2 & & & & & & & \\
\dfrac{q_2}{p_3} & p_3 & & & & & & \\
-\dfrac{1}{q_3} & \dfrac{q_3}{p_4} & p_4 & & & & & \\
0 & -\dfrac{1}{q_4} & \dfrac{q_4}{p_5} & p_5 & & & & \\
\dots & \dots & \dots & \dots & \dots & & & \\
0 & 0 & 0 & 0 & \dots & p_{r-2} & & \\
0 & 0 & 0 & 0 & \dots & \dfrac{q_{r-2}}{p_{r-1}} & p_{r-1} & \\
0 & 0 & 0 & 0 & \dots & -\dfrac{1}{q_{r-1}} & \dfrac{q_{r-1}}{p_r} & p_r
\end{matrix} \right\rangle,$$

$$C_r = \left\langle \begin{matrix}
p_2 & & & & & & & & \\
\dfrac{q_2}{p_3} & p_3 & & & & & & & \\
-\dfrac{1}{q_3} & \dfrac{q_3}{p_4} & p_4 & & & & & & \\
0 & -\dfrac{1}{q_4} & \dfrac{q_4}{p_5} & p_5 & & & & & \\
\dots & \dots & \dots & \dots & \dots & \dots & & & \\
0 & 0 & 0 & 0 & \dots & \dfrac{q_{r-3}}{p_{r-2}} & p_{r-2} & & \\
0 & 0 & 0 & 0 & \dots & -\dfrac{1}{q_{r-2}} & \dfrac{q_{r-2}}{p_{r-1}} & p_{r-1} & \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & -\dfrac{1}{q_r} & q_r
\end{matrix} \right\rangle. \tag{20}$$

As is well known, the rational truncations of continued fractions give the best rational approximations to an irrational number from all fractions with denominators that do not exceed Q_r. This fact follows from the inequality

$$\left|\frac{P_r}{Q_r} - \alpha\right| < \frac{1}{Q_r^2}.$$

We now find such values of p_i that the paradeterminant B_r for an arbitrary value of r, $r \geq 2$, be equal to unity. It is easy to show that

$$p_2 = 1, \qquad p_3 = q_2 + 1, \qquad p_4 = q_3 + 2.$$

We expand the paradeterminant B_r by elements of the last row:

$$B_r = p_r B_{r-1} - q_{r-1} B_{r-2} - B_{r-3}.$$

Taking $B_i = 1$, $i < r$, we easily prove that

$$p_r = q_{r-1} + 2.$$

Hence, the following proposition is true:

Theorem 8. *Suppose that*

$$p_2 = 1, \qquad p_3 = q_2 + 1, \qquad p_i = q_{i-1} + 2, \qquad i = 4, 5, \ldots,$$

in the recursive fraction (19). Then

$$B_i = 1, \qquad i = 1, 2, \ldots,$$

$$C_i = q_i + 1, \qquad i = 3, 4, \ldots,$$

i.e., the equalities

$$\frac{P_r}{Q_r} - \frac{P_{r-1}}{Q_{r-1}} = \frac{(-1)^{r-2}}{Q_r Q_{r-1}},$$

$$\frac{P_r}{Q_r} - \frac{P_{r-2}}{Q_{r-2}} = \frac{(-1)^{r-1}(q_r + 1)}{Q_r Q_{r-2}}$$

are true.

Corollary 2. *For the third-order recursive fraction*

$$\left[\begin{array}{c|cccccc} q_1 & & & & & & \\ \frac{1}{q_2} & q_2 & & & & & \\ \frac{1}{q_2+1} & \frac{q_2+1}{q_3} & q_3 & & & & \\ 0 & \frac{1}{q_3+2} & \frac{q_3+2}{q_4} & q_4 & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & & \\ 0 & 0 & \cdots & \frac{1}{q_{r-1}+2} & \frac{q_{r-1}+2}{q_r} & q_r & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right]_\infty \tag{21}$$

the numerator and denominator of whose rational truncations satisfy the recursive equalities

$$P_k = q_k P_{k-1} + (q_{k-2}+2)P_{k-2} + P_{k-3},$$

$$Q_k = q_k Q_{k-1} + (q_{k-2}+2)Q_{k-2} + Q_{k-3} \tag{22}$$

with initial conditions

$$P_1 = q_1, \quad P_2 = q_1 q_2 + 1, \quad P_3 = q_1 q_2 q_3 + q_1 q_2 + q_1 + q_3 + 1, \tag{23}$$

$$Q_1 = 1, \quad Q_2 = q_2, \quad Q_3 = q_2 q_3 + q_2 + 1,$$

the equalities

$$\frac{P_r}{Q_r} - \frac{P_{r-1}}{Q_{r-1}} = \frac{(-1)^{r-2}}{Q_{r-1}Q_r}, \qquad r = 2,3,\ldots, \tag{24}$$

$$\frac{P_r}{Q_r} - \frac{P_{r-2}}{Q_{r-2}} = \frac{(-1)^{r-1}(q_r+1)}{Q_r Q_{r-2}}, \qquad r = 3,4,\ldots, \tag{25}$$

are true.

This corollary follows directly from Theorem 8.

Proposition 1. *All rational truncations of the recursive fraction (21) are irreducible fractions, i.e.,* $(P_r, Q_r) = 1$ *for all* r, $r > 1$.

Proof. For $r = 1$, we have $P_1 = q_1$, $Q_1 = 1$, and, therefore, $(P_1, Q_1) = 1$. Let $r > 1$. We denote $d = (P_r, Q_r)$. Since $d \mid P_r$ and $d \mid Q_r$, we have

$$d\,|(P_rQ_{r-1}-Q_rP_{r-1}) \quad \text{and} \quad d\,|(-1)^{r-2}, \qquad r = 2,3,\ldots.$$

Hence, $d=1$.

Since q_i, $i=1,2,\ldots$, are natural numbers, it is easy to prove with the use of equalities (24) and (25) that the rational truncations of the recursive fraction (21) are two-sided approximations to its value. The proof is similar to the corresponding proof for continued fractions (see [1], Theorems 4.3 and 4.6).

Let the equalities $q_1=q_2=\ldots=q$ be satisfied in Corollary 2. Then the recursive fraction (21) becomes a 1-periodic recursive fraction of the form

$$x^* = \left[\begin{array}{c|ccccccc} q & & & & & & & \\ \frac{1}{q} & q & & & & & & \\ \frac{1}{q+1} & \frac{q+1}{q} & q & & & & & \\ 0 & \frac{1}{q+2} & \frac{q+2}{q} & q & & & & \\ \cdots & \cdots & \cdots & \cdots & \ddots & & & \\ 0 & 0 & 0 & 0 & \cdots & q & & \\ 0 & 0 & 0 & 0 & \cdots & \frac{q+2}{q} & q & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots \end{array}\right]_\infty . \qquad (26)$$

After several expansions of the numerator and denominator of rational truncation of the recursive fraction (26) by elements of the first column, this quotient can be represented as a fraction with the ratios

$$\frac{\|\cdot\|_{n-1}}{\|\cdot\|_{n-2}}, \qquad \frac{\|\cdot\|_{n-2}}{\|\cdot\|_{n-3}}$$

of triangular matrices of the corresponding orders having the form

$$\left\|\begin{array}{cccccc} q & & & & & \\ \frac{q+2}{q} & q & & & & \\ \frac{1}{q+2} & \frac{q+2}{q} & q & & & \\ 0 & \frac{1}{q+2} & \frac{q+2}{q} & q & & \\ 0 & 0 & \frac{1}{q+2} & \frac{q+2}{q} & q & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \ddots \end{array}\right\|.$$

If r tends to infinity, we obtain after simplifications

$$x^* = \lim_{r\to\infty} \frac{P_r}{Q_r} = q + \frac{1}{q + \frac{q+1}{x} + \frac{1}{x^2}} + \frac{1}{qx + q + 1 + \frac{1}{x}} = q + \frac{x^2 + x}{qx^2 + (q+1)x + 1},$$

where x is the positive root of the equation $x^3 = qx^2 + (q+2)x + 1$, i.e.,

$$x = \frac{q + 1 + \sqrt{(q+1)^2 + 4}}{2}.$$

Thus, we have established the value of the 1-periodic recursive fraction (26):

$$x^* = \frac{q^2 + q + 1}{2} - \frac{(q^2 + q - 1)\sqrt{(q+1)^2 + 4}}{2(q+1)}.$$

Similar investigations for the 2-periodic recursive fraction

$$x^* = \left[\begin{array}{c|ccccc} q_1 & & & & & \\ \frac{1}{q_2} & q_2 & & & & \\ \frac{1}{q_2+1} & \frac{q_2+1}{q_1} & q_1 & & & \\ 0 & \frac{1}{q_1+2} & \frac{q_1+2}{q_2} & q_2 & & \\ 0 & 0 & \frac{1}{q_2+2} & \frac{q_2+2}{q_1} & q_1 & \\ \dots & \dots & \dots & \dots & \dots & \ddots \end{array}\right]_\infty,$$

which can be obtained with the use of Corollary 2 for $q_1 = q_3 = q_5 = \dots$ and $q_2 = q_4 = q_6 = \dots$, lead to the equality

$$x^* = q_1 + \frac{1}{q_2 + \frac{q_2+1}{x} + \frac{1}{xy}} + \frac{1}{q_2 + 1 + q_2 x + \frac{1}{y}}.$$

Here,

$$x = \frac{a + \sqrt{b}}{c}, \qquad y = \frac{\alpha + \sqrt{\beta}}{\gamma},$$

$$a = q_1^2 q_2 + q_1^2 + 2q_1 q_2 + 4q_1 - q_2 + 1,$$

$$c = 2q_1 q_2 + 4q_2 + 2,$$

$$b = q_1^4 q_2^2 + 4q_1^3 q_2^2 + 2q_1^4 q_2 + q_1^4 + 12q_1^3 q_2 + 6q_1^2 q_2^2 + 8q_1^3 + 24q_1^2 q_2 + 18q_1^2$$

$$+ 4q_1 q_2^2 + 20q_1 q_2 + 16q_1 + q_2^2 + 6q_2 + 5,$$

and x is the root of the cubic equation

$$(q_1q_2 + 2q_2 + 1)x^3 - (q_1^2 q_2 + q_1 q_2 + q_1^2 + 4q_1 - 3q_2)x^2 - (q_1^2 q_2 + 3q_1 q_2$$

$$+ q_1^2 + 6q_1 - q_2 + 2)x - (1 + q_1 q_2 + 2q_1) = 0.$$

To find y, we have an analogous cubic equation, and the values of α, β, γ can be calculated by relations similar to the formulas for determining a, b, c, respectively, where q_1 and q_2 should be interchanged. Thus, it is impossible to reach the best rational approximations to cubic irrationalities with the help of 1-periodic and 2-periodic recursive fractions [4].

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