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ROBUST STABILITY AND EVALUATION OF THE QUALITY FUNCTIONAL FOR LINEAR CONTROL SYSTEMS WITH MATRIX UNCERTAINTY

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Summary. *New methods of robust stability analysis for equilibrium states and optimization of linear dynamic systems are developed. Sufficient stability conditions of the zero state are formulated for a linear control systems with uncertain coefficient matrices and measurable output feedback. In addition, a general quadratic Lyapunov function and ellipsoidal set of stabilizing matrices for the feedback amplification coefficients are given. Application of the results is reduced to solving the systems of linear matrix inequalities.*

Key words: *control system, output feedback, robust stability, matrix uncertainty, ellipsoid.*

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Statement of the problem. In applied problems of analysis and synthesis of real objects, one often uses systems of differential and difference equations with uncertain components (parameters, functions and random perturbation) (see, e.g., [1]–[4]). This focuses on the analysis and achievement of performance index of such systems particularly robust stability and optimality.

As set robust stability of dynamic systems we mean parametric or functional set characterizing uncertainty of the given structure of the system and its control components. In particular, in the uncertain linear models matrices of coefficients and feedback may belong to some given sets in the corresponding spaces (intervals, polytopes, affine and ellipsoidal families of matrices, etc.).

The problem of robust stabilization of the control system is to build a static or dynamic control to ensure the asymptotic stability for equilibrium states of the closed-loop system with arbitrary values of uncertain components. Typically, this problem is reduced to solving systems of linear matrix inequalities (LMI).

Analysis of the available investigations. In numerous works find sufficient stability conditions for linear controllable systems with uncertain matrices of coefficients and feedback with respect to measurable output in terms of linear matrix inequalities [3], [5], [6]. A survey of problems and known methods of robust stability analysis and stabilization of feedback control systems can be found in [7]–[9].

The Objective of the work is to develop new methods of robust stability analysis for equilibrium states and optimization of linear difference systems with limited at a norm of matrix uncertainties and static measurable output feedback.

Robust stabilization of control systems. Consider a continuous linear dynamical control system:

$$\dot{x} = (A + \Delta A(t))x + (B + \Delta B(t))u, \quad y = Cx + Du, \tag{1}$$

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where $x \in \mathrm{P}^n$, $u \in \mathrm{P}^m$ and $y \in \mathrm{P}^l$ are state, control, and observable object output vectors respectively, A, B, C and D are constant matrices of corresponding sizes $n \times n$, $n \times m$, $l \times n$ and $l \times m$, and $\Delta A(t) = F_A \Delta_A(t) H_A$, $\Delta B(t) = F_B \Delta_B(t) H_B$ are system uncertainties, where F_A, F_B, H_A, H_B are constant matrices of corresponding size and matrices uncertainties $\Delta_A(t)$ and $\Delta_B(t)$ satisfy the constraints

$$\|\Delta_A(t)\| \le 1,\ \|\Delta_B(t)\| \le 1 \text{ or } \|\Delta_A(t)\|_F \le 1,\ \|\Delta_B(t)\|_F \le 1,\ t \ge 0 .$$

Hereinafter, $\|\cdot\|$ is Euclidean vector norm and spectral matrix norm, $\|\cdot\|_F$ is matrix Frobenius norm, I_n is the unit $n \times n$ matrix, $X = X^{\mathrm{T}} > 0$ (≥ 0) is a positive (nonnegative) definite symmetric matrix. To simplify the records of the matrices dependency on t we will omit.

We control the system (1) with output feedback:

$$u = Ky,\ K = K_0 + \tilde{K},\ \tilde{K} \in \mathrm{E}, \tag{2}$$

where E is an ellipsoidal set of matrices

$$\mathrm{E} = \left\{ K \in \square^{m \times l} : K^{\mathrm{T}} P K \le Q \right\}, \tag{3}$$

where $P = P^{\mathrm{T}} > 0$ and $Q = Q^{\mathrm{T}} > 0$ are symmetric positive definite matrices of corresponding sizes $m \times m$ and $l \times l$.

According to (1)–(3), the following inequality must hold:

$$\left[x^{\mathrm{T}}, u^{\mathrm{T}} \right] \begin{bmatrix} C^{\mathrm{T}} Q C - C^{\mathrm{T}} K_0^{\mathrm{T}} P K_0 C & C^{\mathrm{T}} Q D + C^{\mathrm{T}} K_0^{\mathrm{T}} P G \\ D^{\mathrm{T}} Q C + G^{\mathrm{T}} P K_0 C & \Delta \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \ge 0 ,$$

where $\Delta = D^{\mathrm{T}} Q D - G^{\mathrm{T}} P G$, $G = I_m - K_0 D$. We assume that

$$\Delta < 0 . \tag{4}$$

Then iз $x = 0$ implies $u = 0$, and $x \equiv 0$ is an equilibrium state for the system.

The problem is to construct conditions under which the zero state of the closed-loop control system (1) and (2) is Lyapunov asymptotically stable for every matrix $\tilde{K} \in \mathrm{E}$. The set of stabilizing controls chosen with ellipsoid

$$\mathrm{E}_0 = \left\{ K \in \square^{m \times l} : (K - K_0)^{\mathrm{T}} P (K - K_0) \le Q \right\},$$

e.g., in case when the zero state of the system (1) without control ($u = 0$) is unstable. It is equivalent to choosing a matrix $K = K_0 + \tilde{K}$, $\tilde{K} \in \mathrm{E}$. Matrix K_0 is chosen for the purposes of stabilization the system

$$\dot{x} = M_0 x,\ M_0 = A + \Delta A + (B + \Delta B)(I_m - K_0 D)^{-1} K_0 C . \tag{5}$$

Matrix K_0 can be obtained with methods described in [5].

Note that

$$\|K\| = \sqrt{\lambda_{\max}(K^{\mathrm{T}}K)} \le \rho = \sqrt{\frac{\lambda_{\max}(Q)}{\lambda_{\min}(P)}}\,, \qquad (6)$$

because $\lambda_{\min}(P)K^{\mathrm{T}}K \le K^{\mathrm{T}}PK \le Q \le \lambda_{\max}(Q)I_l$. The quantity ρ in (6) defined by an ellipsoid E is called the *stability radius* of system (1).

We introduce on the set of matrices $\mathrm{K}_D = \{K : \det(I_m - KD) \neq 0\}$ a nonlinear operator

$$\mathrm{D}: \mathbb{R}^{m\times l} \to \mathbb{R}^{m\times l},\ \mathrm{D}(K) = (I_m - KD)^{-1}K \equiv K(I_l - DK)^{-1}.$$

For the operator D the property is performed [5]: if $K_1 \in \mathrm{K}_D$, $K_2 \in \mathrm{K}_D$ and $K_3 = (I_m - K_1D)^{-1}K_2 \in \mathrm{K}_D$ then

$$K_1 + K_2 \in \mathrm{K}_D \text{ and } \mathrm{D}(K_1 + K_2) \equiv \mathrm{D}(K_1) + \mathrm{D}(K_3)\left[I_l + D\mathrm{D}(K_1)\right]. \qquad (7)$$

Under assumption (4) matrix G must be nondegenerate. Therefore values of the operator $\mathrm{D}(K_0) = (I_m - K_0D)^{-1}K_0$ are defined. If $\tilde{K} \in \mathrm{E}$ then values of $\mathrm{D}(K)$ and $\mathrm{D}(K)$ are also defined, where $K = G^{-1}\tilde{K}$. Indeed, under conditions (2) and (4) we have

$$D^{\mathrm{T}}\tilde{K}^{\mathrm{T}}P\tilde{K}D \le D^{\mathrm{T}}QD < G^{\mathrm{T}}PG\,,\ F^{\mathrm{T}}PF < P\,,$$

where $F = \tilde{K}DG^{-1}$ and $P > 0$. Therefore spectral radius $\rho(F) < 1$, and matrix $I_m - F$ is nondegenerate, and hence matrices $I_m - KD = (I_m - F)G$ and $I_m - \hat{K}D = G^{-1}(I_m - KD)$ are nondegenerate as well.

Thus we exclude a control vector from relations (1) and (2) with restriction (4) and we get system

$$\dot{x} = Mx\,,\ M = A + \Delta A + (B + \Delta B)\mathrm{D}(K)C\,. \qquad (8)$$

Separately the zero equilibrium state of system (5) for $K = K_0$ should be asymptotically stable.

Using following statements, we will receive a solution of the formulated problem by means of methods of quadratic Lyapunov function.

Lemma 1. [5] *Suppose that the following system of matrix inequalities hold:*

$$D^{\mathrm{T}}QD + R - P < 0,\ \begin{bmatrix} W & U^{\mathrm{T}} & V^{\mathrm{T}} \\ U & R-P & D^{\mathrm{T}} \\ V & D & -Q^{-1} \end{bmatrix} \le 0\ (< 0)\,, \qquad (9)$$

where $P = P^{\mathrm{T}} > 0$, $Q = Q^{\mathrm{T}} > 0$, $R = R^{\mathrm{T}} \ge 0$, $W = W^{\mathrm{T}} \ge 0$, U, V *and* D *are matrices of suitable sizes. Then for every matrix* $K \in \mathrm{E}$ *the matrix inequality holds:*

$$W + U^{\mathrm{T}} \mathrm{D}(K) V + V^{\mathrm{T}} \mathrm{D}^{\mathrm{T}}(K) U + V^{\mathrm{T}} \mathrm{D}^{\mathrm{T}}(K) R \mathrm{D}(K) V \le 0 \ (< 0) .$$

Lemma 2. [10] *Suppose that* L *is symmetric matrix, the matrix* $M_1, \ldots, M_r$ *and* $N_1, \ldots, N_r$ *have corresponding sizes. Then, if for some numbers* $\varepsilon_1, \ldots, \varepsilon_r > 0$ *matrix inequality*

$$L + \sum_{i=1}^{r} \left(\varepsilon_i M_i M_i^{\mathrm{T}} + \frac{1}{\varepsilon_i} N_i^{\mathrm{T}} N_i \right) \le 0$$

holds, then the inequality

$$L + \sum_{i=1}^{r} \left(M_i \Delta_i N_i + \left(M_i \Delta_i N_i \right)^{\mathrm{T}} \right) \le 0 ,$$

is true for all $\|\Delta_i\| \le 1$ *or* $\|\Delta_i\|_F \le 1$, $i = 1, \ldots, r$.

We will note that Lemmas 1 and 2 are generalizations of the sufficiency statement of the adequacy criterion called Petersen's lemma on matrix uncertainty [11].

Theorem 1. *Suppose that for a positive definite matrix* $X = X^{\mathrm{T}} > 0$ *and for some* $\varepsilon_1, \varepsilon_2 > 0$ *the matrix inequalities (4) and*

$$\begin{bmatrix} \Omega & XB + \varepsilon_2 C^{\mathrm{T}} \mathrm{D}^{\mathrm{T}}(K_0) H_B^{\mathrm{T}} H_B & C_0^{\mathrm{T}} & XF_A & XF_B \\ B^{\mathrm{T}} X + \varepsilon_2 H_B^{\mathrm{T}} H_B \mathrm{D}(K_0) C & -G^{\mathrm{T}} P G + \varepsilon_2 H_B^{\mathrm{T}} H_B & D^{\mathrm{T}} & 0 & 0 \\ C_0 & D & -Q^{-1} & 0 & 0 \\ F_A^{\mathrm{T}} X & 0 & 0 & -\varepsilon_1 I & 0 \\ F_B^{\mathrm{T}} X & 0 & 0 & 0 & -\varepsilon_2 I \end{bmatrix} < 0 \qquad (10)$$

holds, where $\Omega = (A + B\mathrm{D}(K_0)C)^{\mathrm{T}} X + X(A + B\mathrm{D}(K_0)C) + \varepsilon_1 H_A^{\mathrm{T}} H_A + \varepsilon_2 C^{\mathrm{T}} \mathrm{D}^{\mathrm{T}}(K_0) H_B^{\mathrm{T}} H_B \mathrm{D}(K_0) C$. *Then any control (2) ensures asymptotic stability of the zero state for system (1) and the general Lyapunov function* $v(x) = x^{\mathrm{T}} X x$.

Proof. We construct the Lyapunov function for the closed-loop system (8) as $v(x) = x^{\mathrm{T}} X x$. According to Lyapunov's theorem the matrix inequality $X = X^T > 0$ and negative definite derivative of the given function due to system (8) ensure asymptotic stability of the zero equilibrium state, that is with (2) it suffices that the following matrix inequality holds:

$$(A + \Delta A + (B + \Delta B)\mathrm{D}(K_0 + \tilde{K})C)^{\mathrm{T}} X + X(A + \Delta A + (B + \Delta B)\mathrm{D}(K_0 + \tilde{K})C) < 0 . \qquad (11)$$

Using property (7) of operator $\mathrm{D}(K) = (I_m - KD)^{-1} K$, we rewrite inequality (11) as

$$(A + \Delta A)^{\mathrm{T}} X + X(A + \Delta A) + C^{\mathrm{T}} (\mathrm{D}^{\mathrm{T}}(K_0) + (I_l + \mathrm{D}^{\mathrm{T}}(K_0) D^{\mathrm{T}}) \mathrm{D}^{\mathrm{T}}(K))(B + \Delta B)^{\mathrm{T}} X +$$
$$+ X(B + \Delta B)(\mathrm{D}(K_0) + \mathrm{D}(K)(I_l + D\mathrm{D}(K_0)))C < 0 .$$

We rewrite last inequality as

$$M_0^{\mathrm{T}} X + XM_0 + C_0^{\mathrm{T}} \mathrm{D}^{\mathrm{T}}(K)(B+\Delta B)^{\mathrm{T}} X + X(B+\Delta B)\mathrm{D}(K)C_0 < 0 ,$$

where $M_0 = A + \Delta A + (B+\Delta B)\mathrm{D}(K_0)C$, $K = G^{-1}\tilde{K}$. Here

$$\tilde{K} \in \mathrm{E} \Leftrightarrow K \in \mathrm{E} = \left\{ K : K^{\mathrm{T}} PK \le Q \right\} , \qquad (12)$$

where $P = G^{\mathrm{T}} PG$.

We use Lemma 1 putting $W = M_0^{\mathrm{T}} X + XM_0$, $U = (B+\Delta B)^{\mathrm{T}} X$, $V = C_0$, $R = 0$. Inequality (4) follows from the first block inequality in (9). Then the second block inequality in (9) has the form

$$\begin{bmatrix} M_0^{\mathrm{T}} X + XM_0 & X(B+\Delta B) & C_0^{\mathrm{T}} \\ (B+\Delta B)^{\mathrm{T}} X & -G^{\mathrm{T}} PG & D^{\mathrm{T}} \\ C_0 & D & -Q^{-1} \end{bmatrix} \le 0 .$$

Using the structure of matrix uncertainties $\Delta_A(t)$, $\Delta_B(t)$, we decompose the last inequality:

$$\begin{bmatrix} A^{\mathrm{T}} X + XA + C^{\mathrm{T}} \mathrm{D}^{\mathrm{T}}(K_0) B^{\mathrm{T}} X + XB\mathrm{D}(K_0)C & XB & C_0^{\mathrm{T}} \\ B^{\mathrm{T}} X & -G^{\mathrm{T}} PG & D^{\mathrm{T}} \\ C_0 & D & -Q^{-1} \end{bmatrix} + \begin{bmatrix} H_A^{\mathrm{T}} \\ 0 \\ 0 \end{bmatrix} \Delta_A^{\mathrm{T}} \begin{bmatrix} F_A^{\mathrm{T}} X & 0 & 0 \end{bmatrix} +$$

$$+ \begin{bmatrix} XF_A \\ 0 \\ 0 \end{bmatrix} \Delta_A \begin{bmatrix} H_A & 0 & 0 \end{bmatrix} + \begin{bmatrix} C^{\mathrm{T}} \mathrm{D}^{\mathrm{T}}(K_0) H_B^{\mathrm{T}} \\ 0 \\ 0 \end{bmatrix} \Delta_B^{\mathrm{T}} \begin{bmatrix} F_B^{\mathrm{T}} X & 0 & 0 \end{bmatrix} + \begin{bmatrix} XF_B \\ 0 \\ 0 \end{bmatrix} \Delta_B \begin{bmatrix} H_B \mathrm{D}(K_0)C & 0 & 0 \end{bmatrix} +$$

$$+ \begin{bmatrix} 0 \\ H_B^{\mathrm{T}} \\ 0 \end{bmatrix} \Delta_B^{\mathrm{T}} \begin{bmatrix} F_B^{\mathrm{T}} X & 0 & 0 \end{bmatrix} + \begin{bmatrix} XF_B \\ 0 \\ 0 \end{bmatrix} \Delta_B \begin{bmatrix} 0 & H_B & 0 \end{bmatrix} < 0 ,$$

which is done for Lemma 2 if there are $\varepsilon_1, \varepsilon_2 > 0$ such as

$$\begin{bmatrix} A^{\mathrm{T}} X + XA + C^{\mathrm{T}} \mathrm{D}^{\mathrm{T}}(K_0) B^{\mathrm{T}} X + XB\mathrm{D}(K_0)C & XB & C_0^{\mathrm{T}} \\ B^{\mathrm{T}} X & -G^{\mathrm{T}} PG & D^{\mathrm{T}} \\ C_0 & D & -Q^{-1} \end{bmatrix} +$$

$$+\varepsilon_1 \begin{bmatrix} H_A^{\mathrm{T}} H_A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{\varepsilon_1} \begin{bmatrix} XF_A F_A^{\mathrm{T}} X & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} +$$

$$+\varepsilon_2 \begin{bmatrix} C^{\mathrm{T}} \mathrm{D}^{\mathrm{T}}(K_0) H_B^{\mathrm{T}} H_B \mathrm{D}(K_0) C & C^{\mathrm{T}} \mathrm{D}^{\mathrm{T}}(K_0) H_B^{\mathrm{T}} H_B & 0 \\ H_B^{\mathrm{T}} H_B \mathrm{D}(K_0) C & H_B^{\mathrm{T}} H_B & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{\varepsilon_2} \begin{bmatrix} XF_B F_B^{\mathrm{T}} X & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \le 0 .$$

We use the well-known criterion of nonpositive (negative) definite of block matrices for the last inequality (Schur's lemma [12]): if $\det V \neq 0$ then

$$\begin{bmatrix} U & Z \\ Z^{\mathrm{T}} & V \end{bmatrix} \leq 0 \ (<0) \ \Leftrightarrow \ V<0, \ U - ZV^{-1}Z^{\mathrm{T}} \leq 0 \ \ (<0).$$

We get inequality equivalent to condition of the form (10) under which together with (4) matrix inequality (11) holds. These conditions ensure asymptotic stability for the zero state of the closed-loop system (8) for any control (2).

This completes the proof of the theorem.

Bounds on the quadratic quality criterion under uncertainty conditions. Consider a control system (1), (2) with quadratic quality functional

$$J(u, x_0) = \int_0^{\infty} \varphi(x,u)dt, \ \ \varphi(x,u) = \begin{bmatrix} x^{\mathrm{T}} & u^{\mathrm{T}} \end{bmatrix} \Phi \begin{bmatrix} x \\ u \end{bmatrix}, \ \Phi = \begin{bmatrix} S & N \\ N^T & R \end{bmatrix} > 0, \tag{13}$$

where x_0 is initial vector, $S = S^{\mathrm{T}} > 0$, $R = R^{\mathrm{T}} > 0$, and N given constant matrices.

We need to describe the set of controls (2) that would provide asymptotic stability for the state $x \equiv 0$ of system (1) and a bound

$$J(u, x_0) \leq \omega, \tag{14}$$

where $\omega > 0$ is some maximal admissible value of the functional. When solving this problem, we still use the quadratic Lyapunov function $v(x) = x^{\mathrm{T}} Xx$ under constraint

$$x_0^{\mathrm{T}} X x_0 \leq \omega. \tag{15}$$

Under assumptions (2) and (4) values of $\mathrm{D}(K)$, $\mathrm{D}(K_0)$, and $\mathrm{D}(K)$ are defined, where $K = G^{-1}\tilde{K}$, $G = I_m - K_0 D$. Here the closed-loop system can be represented as (8), and the derivative of function $v(x)$ due to system (8) and the expression under the integral in (13) have the form

$$\dot{v}(x) = x^{\mathrm{T}}(M^{\mathrm{T}}X + XM)x, \ \ \varphi(x,u) = x^{\mathrm{T}} L^{\mathrm{T}} \Phi L x,$$

where $L^{\mathrm{T}} = \begin{bmatrix} I_n & C^{\mathrm{T}} \mathrm{D}^{\mathrm{T}}(K) \end{bmatrix}$, $K = K_0 + \tilde{K}$.

We now require that together with (4) the following inequality holds:

$$\dot{v}(x) \leq -\varphi(x,u). \tag{16}$$

For this it suffices that the following matrix inequality holds:

$$M^{\mathrm{T}}X + XM + L^{\mathrm{T}}\Phi L < 0. \tag{17}$$

Then the zero solution $x \equiv 0$ of system (8) is asymptotically stable and together with (16) we get an upper bound on the functional:

$$J(u,x_0) \le -\int_0^\infty \frac{d}{dt} v(x)dt = x_0^{\mathrm{T}} X x_0 \le \omega \,, \tag{18}$$

Using property (7) of operator D, we rewrite inequality (17) as

$$W + U^{\mathrm{T}} \mathrm{D}(K)V + V^{\mathrm{T}} \mathrm{D}^{\mathrm{T}}(K)U + V^{\mathrm{T}} \mathrm{D}^{\mathrm{T}}(K) R \mathrm{D}(K)V < 0 \,, \tag{19}$$

where $W = M_0^{\mathrm{T}} X + X M_0 + L_0^{\mathrm{T}} \Phi L_0$, $U = (B + \Delta B)^{\mathrm{T}} X + N^{\mathrm{T}} + R\mathrm{D}(K_0)C$, $V = C_0$, $L_0^{\mathrm{T}} = \left[I_n \quad C^{\mathrm{T}} \mathrm{D}^{\mathrm{T}}(K_0) \right]$. Here condition (12) hold.

Applying Lemma 1, relations (16)–(19), and Lemma 2, we arrive at the following result.

Theorem 2. *Suppose that for a positive definite matrix* $X = X^T > 0$ *and for some* $\varepsilon_1, \varepsilon_2 > 0$ *the matrix inequalities (15) and*

$$G^{\mathrm{T}} P G - D^{\mathrm{T}} Q D > R, \tag{20}$$

$$\begin{bmatrix} Z & N_0 & C_0^{\mathrm{T}} & XF_A & XF_B \\ N_0^{\mathrm{T}} & R - G^{\mathrm{T}}PG & D^T & 0 & 0 \\ C_0 & D & Q^{-1} & 0 & 0 \\ F_A^{\mathrm{T}} X & 0 & 0 & -\varepsilon_1 I & 0 \\ F_B^{\mathrm{T}} X & 0 & 0 & 0 & -\varepsilon_2 I \end{bmatrix} < 0 \,, \tag{21}$$

holds, where $Z = (A + B\mathrm{D}(K_0)C)^{\mathrm{T}} X + X(A + B\mathrm{D}(K_0)C) + L_0^{\mathrm{T}} \Phi L_0 + \varepsilon_1 H_A^{\mathrm{T}} H_A + \varepsilon_2 C_*^{\mathrm{T}} C_*$, $N_0 = XB + N + C^{\mathrm{T}} \mathrm{D}^{\mathrm{T}}(K_0) R + \varepsilon_2 C_*^{\mathrm{T}} H_B$, $C_* = H_B \mathrm{D}(K_0) C$. *Then any control (2) ensures asymptotic stability of the zero state for system (1), the general Lyapunov function* $v(x) = x^{\mathrm{T}} X x$, *and a bound on the functional (14).*

Based on Theorem 2 and its corollaries, we can formulate the following optimization problem for system (1): *minimize* $\omega > 0$ *under constraints* (15), (20) and (21).

The results of Theorems 1–2 can be generalized in case when

$$\Delta A(t) = \sum_{i=1}^{r} F_A^i \Delta_A^i(t) H_A^i \,, \; \Delta B(t) = \sum_{i=1}^{r} F_B^i \Delta_B^i(t) H_B^i \,.$$

Numerical experiment. Consider a control system for a double oscillator. It is system of two solids that connected by a spring and slide without a friction along of horizontal rod (Fig. 1). This system is defined with two linear differential equations of order two, or, in vector-matrix form [13]:

$$\dot{x} = (A + \Delta A(t))x + Bu \,, \tag{22}$$

where

$$A+\Delta A(t)=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{k_0}{m_1} & \dfrac{k_0}{m_1} & 0 & 0 \\ \dfrac{k_0}{m_2} & -\dfrac{k_0}{m_2} & 0 & 0 \end{bmatrix}+F_A\Delta(t)H_A,\ B=\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},\ F_A=\begin{bmatrix} 0 \\ 0 \\ -\delta \\ \delta \end{bmatrix},$$

$$H_A=\begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix},.\ x=\begin{bmatrix} x_1 & x_2 & \dot{x}_1 & \dot{x}_2 \end{bmatrix}^{\mathrm{T}}$$

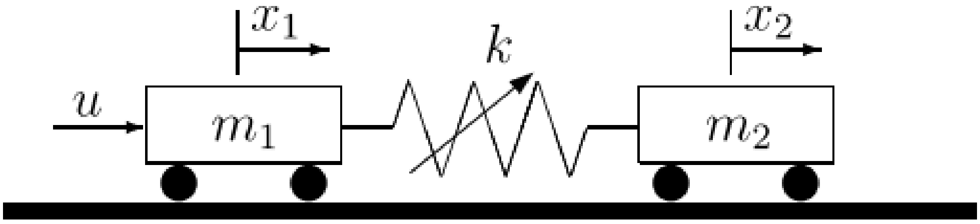

Figure 1. A two-masse mechanical system

Here x_1 and $\dot{x}_1$ are coordinate and velocity respectively for the first solid, x_2 and $\dot{x}_2$ are coordinate and velocity respectively for the second solid, m_1 and m_2 are masses of the first and second solids respectively. We define a stiffness coefficient as variable periodic function of time $k=k_0+\delta\Delta(t)$, where $\Delta(t)=\sin(\varpi t)$, $\delta<<1$ is the amplitude of harmonic oscillations, and ϖ is the frequency parameter.

Let $m_1=1$, $m_2=1$, $k_0=2$, $\delta=0{,}02$, $\Delta(t)=\sin(t/5)$.

We assume that the output vector

$$y=Cx+Du=\begin{bmatrix} \dot{x}_1+u \\ x_2 \end{bmatrix},\ C=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},\ D=\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

can be measured. We find control in the form static output feedback $u=Ky$, where $K=\begin{bmatrix} k_1 & k_2 \end{bmatrix}=K_0+\tilde{K}$. We find the vector $K_0=\begin{bmatrix} 1{,}6938 & 0{,}1089 \end{bmatrix}$ that ensures asymptotic stability for system $\dot{x}=M_0x$, $M_0=A+B\mathrm{D}(K_0)C$. Here the spectrum equals $\sigma(M_0)=\{-0{,}3259\pm1{,}6913i;\ -0{,}8333;-0{,}3296\}$. The behavior of solutions of system with matrix uncertainty (22) with control $u=K_0y$ and initial vector $x_0=\begin{bmatrix} 1 & -2 & 0 & 2 \end{bmatrix}^{\mathrm{T}}$ is shown on Fig. 2.

For demonstration of Theorem 2 we define a matrix functional (13): $S=0{,}1I_4$, $R=0{,}01$, $N^T=\begin{bmatrix} 0{,}01 & 0 & 0 & 0{,}01 \end{bmatrix}$. Using the Matlab suite, we find $P=39{,}5751$ and positive definite matrices

$$Q = \begin{bmatrix} 29,6673 & 11,8521 \\ 11,8521 & 17,0069 \end{bmatrix}, \ X = \begin{bmatrix} 114,3996 & -49,8299 & 42,2629 & 37,6176 \\ -49,8299 & 55,1945 & -15,0903 & -5,6888 \\ 42,2629 & -15,0903 & 36,7322 & 15,6111 \\ 37,6176 & 5,6888 & 15,6111 & 39,8411 \end{bmatrix},$$

that satisfy the inequalities (20), (21) for $\varepsilon_1 = 0{,}01$.

Thus, for all values of the vector of feedback amplification coefficients $K = K_0 + \tilde{K}$ from a closed region E_0 bounded by the ellipse $(K - K_0)Q^{-1}(K - K_0)^{\mathrm{T}} \le P^{-1}$ (Fig. 3), the motion of the system of two solids in a neighborhood of the zero state is asymptotically stable. Here $v(x) = x^{\mathrm{T}} X x$ is a general Lyapunov function, and the value of the given quality functional does not exceed $v(x_0) = 889{,}8436$. Stability radius of system equals $\rho = 0{,}9639$.

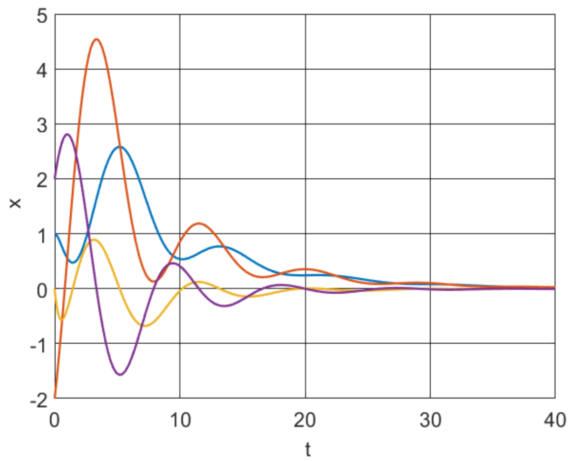

Figure 2. Behavior of the control system $u = K_0 y$

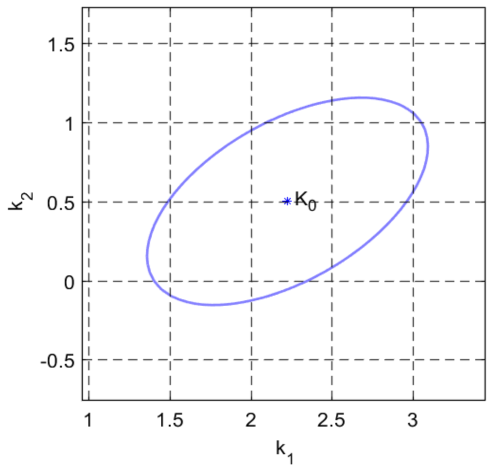

Figure 3. Region of feedback amplification coefficients

Conclusions. In this work, we have proposed new methods of robust stability analysis and optimization of linear difference systems with static output feedback. Here values of unknown matrix coefficients are defined by restrictions on norm of matrix uncertainties and the measurable output vector contains components of both the system state and the control. Practical implementation of the proposed methods is related to solving differential or algebraic LMI. An important characteristic feature that distinguishes LMI that we have found from known ones is the possibility to construct an ellipsoid of stabilizing matrices for the feedback amplification coefficients, general quadratic Lyapunov function, and also bounds on the quadratic quality functional for linear control systems with the considered matrix uncertainties.

The results are obtained based on the known generalizations statement on adequacy of Petersen's lemma about matrix uncertainties. Unfortunately, the conditions of theorem 1-2 are generally theoretical. Their practical use in problems of output robust stabilization based on quadratic Lyapunov functions with uncertain matrices requires special methods of matrix K_0 (see, e.g., [7, 9]). This is one of the topical tasks of the following studies.

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РОБАСТНА СТІЙКІСТЬ ТА ОЦІНЮВАННЯ ФУНКЦІОНАЛА ЯКОСТІ ЛІНІЙНИХ СИСТЕМ КЕРУВАННЯ З МАТРИЧНОЮ НЕВИЗНАЧЕНІСТЮ

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Резюме. *Розроблено нові методи аналізу робастної стійкості стану рівноваги й оптимізації лінійних систем керування зі зворотним зв'язком по виходу. В системах керування присутні матричні невизначеності, значення яких задані обмеженнями по нормі, а вимірний вектор виходу містить компоненти як стану системи, так і керування. Для таких систем сформованано достатні умови стійкості нульового стану рівноваги. Практична реалізація отриманих методів зводиться до розв'язування алгебраїчних лінійних матричних нерівностей. Відмінною особливістю отриманих лінійних матричних нерівностей від відомих є можливість побудови еліпсоїда стабілізуючих матриць коефіцієнтів підсилення зворотного зв'язку, спільної квадратичної функції Ляпунова, а також верхнього оцінювання квадратичного функціонала якості для лінійних систем керування з розглянутими невизначеностями. Результати роботи отримані на основі відомих узагальнень твердження достатності леми Пітерсена про матричні невизначеності. Розглянуто застосування отриманих теорем для стабілізації та оптимізації системи керування двійним осцилятором. Для розв'язування побудованої системи лінійних матричних нерівностей застосовано ефективні засоби LMI Toolbox комп'ютерної системи Matlab. Отримані достатні умови стійкості стану рівноваги і оптимізації лінійних динамічних систем у загальному випадку мають теоретичний характер. Їх практичне використання в задачах робастної стабілізації по виходу на основі побудови квадратичних функцій Ляпунова з невизначеними матрицями потребує спеціальних методів знаходження матриці, яка визначає центр еліпсоїда. Це є однією з актуальних задач наступних досліджень. Отримані результати можуть бути використані при розробленні алгоритмів робастної стабілізації й оптимізації динамічних систем, наприклад із зовнішніми збуреннями.*

Ключові слова: *система керування, зворотний зв'язок, робастна стійкість, матрична невизначеність, еліпсоїд.*

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