INVARIANT SETS AND COMPARISON OF DYNAMICAL SYSTEMS

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We propose a method for the construction and investigation of invariant sets of differential systems described by cone inequalities with the use of the operator of differentiation along the trajectories of the system. Well-known conditions for the positivity of linear and nonlinear differential systems with respect to typical classes of cones are generalized. A method for comparison and ordering is developed for a family of dynamical systems.

Introduction

In practical investigations, one often uses differential and difference models of dynamical objects whose phase spaces contain invariant sets (in particular, cones). The problem of the construction and classification of these sets is one of the most important problems of qualitative analysis of dynamical systems. Invariant sets of systems must be taken into account and used in problems of analysis of stability and control (see, e.g., [1, 2]).

In the present paper, we propose a method for the construction of invariant sets of differential systems in the form of cone inequalities using the operator of differentiation along the trajectories of the system and elements of the conjugate cone. As a corollary, we formulate a generalized comparison principle for a finite family of independent systems. We give several examples of application of the proposed method to first-order and second-order differential systems. Well-known results on the invariance of cones are special cases of the established criterion for the invariance of a given class of sets. In particular, we establish sufficient conditions for the invariance of a time-varying ellipsoidal cone for a certain class of nonlinear differential systems. Analogous results for linear systems were established in [3, 4].

1. Definition and Auxiliary Facts

A convex closed set $\mathcal{K}$ of a real normed space $\mathcal{E}$ is called a wedge if $\alpha\mathcal{K}+\beta\mathcal{K}\subset\mathcal{K}\ \ \forall\alpha,\beta\geq 0.$ A wedge $\mathcal{K}$ with edge $\mathcal{K}\cap-\mathcal{K}=\{0\}$ is a cone. The conjugate cone $\mathcal{K}^*$ is formed by linear functionals $\varphi\in\mathcal{E}^*$ that take nonnegative values on elements of $\mathcal{K};$ furthermore, $\mathcal{K}=\{X\in\mathcal{E}\colon\varphi(X)\geq 0\ \ \forall\varphi\in\mathcal{K}^*\}.$ A space with a wedge is semiordered: $X\leq Y\iff Y-X\in\mathcal{K}.$ A cone $\mathcal{K}$ with a nonempty set of interior points $\operatorname{int}\mathcal{K}=\{X\colon X>0\}$ is solid. A cone $\mathcal{K}$ is called normal if the relation $0\leq X\leq Y$ implies that $\|X\|\leq\nu\|Y\|,$ where ν is a universal constant. The least of these numbers ν is the normality constant of the cone. If $\mathcal{E}=\mathcal{K}-\mathcal{K},$ then the cone $\mathcal{K}$ is reproducing. The cone $\mathcal{K}$ is normal only if the conjugate cone $\mathcal{K}^*$ is reproducing.

Let a cone $\mathcal{K}_1(\mathcal{K}_2)$ be selected in a Banach space $\mathcal{E}_1(\mathcal{E}_2).$ An operator $M\colon\mathcal{E}_1\to\mathcal{E}_2$ is called monotone if the relation $X\geq Y$ implies that $MX\geq MY.$ The monotonicity of a linear operator is equivalent to its positivity: $X\geq 0\implies MX\geq 0.$

A dynamical system whose state $X(t)=\Omega(t,t_0)X_0$ at every time $t>t_0$ is defined by a positive (monotone) operator $\Omega(t,t_0)\colon\mathcal{X}\to\mathcal{X}$ is positive (monotone) with respect to a certain cone. A system has an invariant set

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$\mathcal{K}_t \subset \mathcal{X}$ if, for any $t_0 \geq 0$, the relation $X_0 \in \mathcal{K}_{t_0}$ implies that $X(t) \in \mathcal{K}_t$ for $t \geq t_0$. If $\mathcal{K}_t$ is a cone, then the inequalities between the elements of the space at every time t are denoted by the symbols $\overset{\mathcal{K}_t}{\leq}$ or $\overset{\mathcal{K}_t}{\geq}$.

The fact that the differential system

$$\dot{X} = F(X,t), \quad X \in \mathcal{X}, \ t \geq 0, \tag{1.1}$$

belongs to the indicated classes can be established by using the elements of the conjugate cone. In particular, system (1.1) is positive and monotone with respect to a solid cone $\mathcal{K}_t$ if $t < \tau \Longrightarrow \mathcal{K}_t \subseteq \mathcal{K}_\tau$ and the following conditions [5] are satisfied:

$$X \overset{\mathcal{K}_t}{\geq} 0, \, \varphi \in \mathcal{K}_t^*, \, \varphi(X) = 0 \implies \varphi\left(F(X,t)\right) \geq 0, \tag{1.2}$$

$$X \overset{\mathcal{K}_t}{\leq} Y, \, \varphi \in \mathcal{K}_t^*, \, \varphi(X-Y) = 0 \Longrightarrow \varphi\left(F(Y,t) - F(X,t)\right) \geq 0, \tag{1.3}$$

where $\mathcal{K}_t^*$, $t \geq 0$, is the conjugate cone.

The isolated equilibrium state $X \equiv 0$ of a dynamical system is called stable in $\mathcal{K}_t$ if, for any $\varepsilon > 0$ and $t_0 \geq 0$, one can find $\delta > 0$ such that the relation $X_0 \in \mathcal{S}_\delta(t_0)$ yields $X(t) \in \mathcal{S}_\varepsilon(t)$ for $t > t_0$, where $\mathcal{S}_\varepsilon(t) = \{X \in \mathcal{K}_t \colon \|X\| \leq \varepsilon\}$. Furthermore, if, for a certain $\delta_0 > 0$, the relation $X_0 \in \mathcal{S}_{\delta_0}(t_0)$ implies that $\|X(t)\| \to 0$ as $t \to \infty$, then the state $X \equiv 0$ of the system is asymptotically stable in $\mathcal{K}_t$. If the state $X \equiv 0$ of the system with invariant cone $\mathcal{K}_t$ is Lyapunov-stable (asymptotically stable), then it is stable (asymptotically stable) in $\mathcal{K}_t$.

For dynamical systems with discrete time, the invariant sets and the properties of positivity and monotonicity with respect to a cone and stability in $\mathcal{K}_t$ are defined by analogy.

The triple of numbers $i(S) = \{i_+(S), i_-(S), i_0(S)\}$, where $i_+(S)$, $i_-(S)$, and $i_0(S)$ are the numbers of, respectively, positive, negative, and zero values of S, counting multiplicity, is called the inertia of the symmetric matrix $S = S^T \in R^{n \times n}$.

2. Construction of Invariant Sets in the Phase Space of Differential Systems

In a Banach space, we consider the differential system (1.1), where $F \colon \mathcal{X} \times [0,\infty) \to \mathcal{X}$ is an operator that satisfies conditions for the existence and uniqueness of solutions $X(t)$ in a certain domain $\Omega \subset \mathcal{X}$ with initial conditions $X(t_0) = X_0 \in \Omega$. System (1.1) has an invariant set $\mathcal{I}_t \subset \mathcal{X}$ if the inclusion $X(t_0) \in \mathcal{I}_{t_0}$ implies that $X(t) \in \mathcal{I}_t$ for $t > t_0 \geq 0$.

We construct invariant sets of system (1.1) in the form

$$\mathcal{I}_t = \{X \in \Omega \colon V(X,t) \overset{\mathcal{K}}{\geq} 0\}, \tag{2.1}$$

where $V \colon \mathcal{X} \times [0,\infty) \to \mathcal{E}$ is a certain operator and $\overset{\mathcal{K}}{\geq}$ is the inequality generated by a given cone or wedge $\mathcal{K}$ in the space $\mathcal{E}$. For this purpose, we define the operator D_t of differentiation along the trajectories of the system as the (strong) derivative of a composite function, i.e.,

$$D_t V(X,t) = \frac{d}{d\tau}\, V(\Psi(\tau,t,X),\tau)\big|_{\tau=t}\,, \tag{2.2}$$

where $X(\tau) = \Psi(\tau, t, X)$ is the solution of the system with initial condition $X(t) = X$. We assume that $V(X,t)$ is a continuous function together with its partial derivatives in the domain $\Omega \times [0, \infty)$.

We now give several known relations for the operator D_t using not solutions of system (1.1) but its right-hand side F. For example, if $\mathcal{X} = R^n$ and $\mathcal{E} = R^m$, then

$$D_t V(X,t) = V'_X(X,t)\, F(X,t) + V'_t(X,t),$$

where $V'_X(X,t)$ is an $m \times n$ Jacobi matrix composed of partial derivatives of the function V with respect to X. By analogy, we can consider a generalization of this relation based on the application of the Gâteaux and Fréchet derivatives of a nonlinear operator [6]. For example, we can assume that $V'_t(X,t)$ is the strong derivative of a function with respect to t, and $V'_X(X,t)$ is the Gâteaux derivative with respect to X, i.e., a linear bounded operator of the type

$$V'_X(X,t)H = \frac{d}{d\tau}\, V(X + \tau H, t)|_{\tau=0}\,.$$

Remark 2.1. In the theory of comparison of systems, it is customary to use the upper right and left derivatives along the trajectories of a Dini-type system, namely

$$D_t^{\pm} V(X,t) = \limsup_{\tau \to 0\pm} \frac{1}{\tau}\left[V\left(X + \tau F(X,t), t + \tau\right) - V(X,t)\right],$$

under the condition that the function $V(X,t)$ is not differentiable and is only continuous and locally Lipschitzian with respect to X (see, e.g., [7, 8]).

Theorem 2.1. *Let $\mathcal{K}$ be a solid cone. Then $\mathcal{I}_t$ is an invariant set of system (1.1) if and only if the following condition is satisfied for every $t \geq 0$:*

$$X \in \mathcal{I}_t, \quad \varphi \in \mathcal{K}^*, \quad \varphi\left(V(X,t)\right) = 0 \quad \Longrightarrow \quad \varphi\left(D_t V(X,t)\right) \geq 0. \tag{2.3}$$

Proof. Let $X(t)$ be a solution of system (1.1) with initial condition $X(t_0) = X_0 \in \mathcal{I}_{t_0}$. Then D_t acts as the operator of differentiation of a composite function $V(X(t),t)$ with respect to time and the following equality is true:

$$\int\limits_{t_0}^{t} D_\tau V(X(\tau),\tau)\, d\tau = V(X(t),t) - V(X_0,t_0).$$

In particular, this implies that $V(X(t),t) \overset{\mathcal{K}}{\geq} V(X_0,t_0)$ if $D_t V(X,t) \overset{\mathcal{K}}{\geq} 0$ for $X \in \mathcal{I}_t$ and $t \geq t_0$. Moreover, $V(X(t),t) \overset{\mathcal{K}}{>} 0$ if $V(X_0,t_0) \overset{\mathcal{K}}{>} 0$.

Assume that, at a certain time $\tau \geq t_0$, the value of the function $V(X_\tau, \tau)$, where $X_\tau = X(\tau)$, reaches the boundary of the cone $\mathcal{K}$. Then, for a certain nonzero functional $\varphi \in \mathcal{K}^*$, we have $\varphi(V(X_\tau,\tau)) = 0$.

Together with (2.1), we consider the set

$$\mathcal{I}_t^\varepsilon = \{X \in \Omega\colon V_\varepsilon(X,t) \overset{\mathcal{K}}{\geq} 0\}, \quad V_\varepsilon(X,t) = V(X,t) + \varepsilon\omega(t)Y,$$

where $\varepsilon > 0,\ Y \overset{\mathcal{K}}{>} 0,$ and $\omega(t)$ is a nonnegative continuously differentiable function such that $\omega(\tau) = 0$ and $\dot{\omega}(\tau) > 0.$ We set, e.g., $\omega(t) = \arctan(t-\tau).$ Then it is obvious that $\mathcal{I}_t \subset \mathcal{I}_t^\varepsilon,$ and, furthermore, $\mathcal{I}_t^\varepsilon \to \mathcal{I}_t$ as $\varepsilon \to 0,\ t \geq \tau.$

Since $V_\varepsilon(X_\tau, \tau) = V(X_\tau, \tau)$ and $\varphi(Y) > 0,$ we conclude that, for a certain $\delta > 0,$ according to condition (2.3), the following relations are true:

$$\varphi\left(D_t V_\varepsilon(X,t)\right) = \varphi\left(D_t V(X,t)\right) + \frac{\varepsilon}{1+(t-\tau)^2}\,\varphi(Y) \geq 0, \quad \tau \leq t \leq \tau+\delta,$$

$$\int\limits_{\tau}^{\tau+\delta} \varphi\left(D_t V_\varepsilon(X(t),t)\right)\,dt = \varphi(V_\varepsilon(X(\tau+\delta), \tau+\delta)) \geq 0.$$

This means that, at time $\tau,$ the trajectory $X(t)$ cannot leave the set $\mathcal{I}_\tau^\varepsilon,$ i.e., $V_\varepsilon(X(t),t) \overset{\mathcal{K}}{\geq} 0$ for $\tau \leq t \leq \tau+\delta.$ Otherwise, the opposite inequality $\varphi(V_\varepsilon(X(\tau+\delta), \tau+\delta)) < 0$ must be true for a certain $\varphi \in \mathcal{K}^*$ and an arbitrarily small $\delta > 0.$

By virtue of the closedness of the cone $\mathcal{K},$ as $\varepsilon \to 0$ we get

$$V_\varepsilon(X(t),t) \to V(X(t),t) \overset{\mathcal{K}}{\geq} 0, \quad \tau \leq t \leq \tau+\delta.$$

Thus, $\mathcal{I}_t$ is an invariant set of system (1.1).

The converse statement is a corollary of the Lagrange theorem:

$$\varphi(V(X(\tau+\delta), \tau+\delta)) - \varphi(V(X(\tau),\tau)) = \delta\, \varphi(D_\xi V(X(\xi),\xi)),$$

where $\xi \in (\tau, \tau+\delta).$ If $\varphi(V(X(\tau),\tau)) = 0$ and $X(\tau+\delta) \in \mathcal{I}_{\tau+\delta},$ then it is necessary that the inequality $\varphi(D_\tau V(X(\tau),\tau)) \geq 0$ be true for sufficiently small $\delta > 0.$

The theorem is proved.

Remark 2.2. Condition (2.3) is satisfied if, for a certain continuous scalar function $\alpha(X,t),$ the following cone inequality is true:

$$D_t V(X,t) + \alpha(X,t) V(X,t) \overset{\mathcal{K}}{\geq} 0, \quad X \in \partial\mathcal{I}_t, \quad t \geq 0. \tag{2.4}$$

We give several examples of the application of Theorem 2.1 to the construction of invariant sets and, in particular, cones of the type (2.1) for some classes of systems.

Example 2.1. Consider the nonlinear system

$$\dot{x} = A(x,t)\,x, \quad x \in R^n, \quad t \geq 0. \tag{2.5}$$

We define set (2.1) by using the cone of nonnegative vectors $\mathcal{K} = R_+^n$ and the vector function $V(x,t) = R(t)\,x,$ where $R(t)$ is a nondegenerate continuously differentiable matrix function. Condition (2.4) is satisfied if, for a certain matrix $\alpha(x,t),$ all elements of the matrix

$$B_\alpha(t) = \dot{R}(t) R^{-1}(t) + R(t)[A(x,t) + \alpha(x,t) I] R^{-1}(t)$$

are nonnegative functions. The last restriction reduces to the form

$$b_{ij}(x,t) \geq 0, \quad i \neq j, \quad x \in \partial\mathcal{K}, \quad t \geq 0, \tag{2.6}$$

where $b_{ij}(x,t)$ are elements of the matrix $B_\alpha(t)$ for $\alpha = 0$. In the special case $R(t) \equiv I$, set (2.1) is the cone $\mathcal{K}$, and inequalities (2.6) generalize known conditions for the positivity of linear systems with respect to $\mathcal{K}$ [2].

Example 2.2. Consider the case where set (2.1) is described by the function

$$V(x,t) = x^T P(t)x + q^T(t)x + r(t),$$

where the symmetric matrix $P(t)$, vector function $q(t)$, and scalar function $r(t)$ are continuous and differentiable for $t \geq 0$. Inequality (2.4), which guarantees the invariance of this set for system (2.5), has the form

$$x^T P_\alpha(x,t)x + q_\alpha^T(x,t)x + r_\alpha(x,t) \geq 0, \quad x \in \partial\mathcal{I}_t, \quad t \geq 0, \tag{2.7}$$

where

$$P_\alpha(x,t) = \dot{P}(t) + \alpha(x,t)P(t) + A^T(x,t)P(t) + P(t)A(x,t),$$

$$q_\alpha(x,t) = \dot{q}(t) + \alpha(x,t)q(t) + A^T(x,t)q(t),$$

$$r_\alpha(x,t) = \dot{r}(t) + \alpha(x,t)r(t).$$

In particular, we can require that $P_\alpha(x,t) \geq 0$, $q_\alpha(x,t) \equiv 0$, and $r_\alpha(x,t) \geq 0$. Using these relations, we establish the invariance of set (2.1) in system (2.5).

Example 2.3. For the nonlinear system

$$\dot{x} = f(x,t), \quad x \in R^{n+1}, \quad t \geq 0, \tag{2.8}$$

we construct invariance conditions for the varying ellipsoidal cone $\mathcal{I}_t$ described in the form (2.1) provided that

$$V(x,t) = \begin{bmatrix} x^T Q(t)x \\ h^T(t)x \end{bmatrix}, \quad \mathcal{K} = R_+^2,$$

where $h(t)$ is the eigenvector of a symmetric matrix $Q(t)$ with inertia $i(Q(t)) \equiv \{1, n, 0\}$ corresponding to its unique positive eigenvalue.

We verify condition (2.3), where

$$D_t V(x,t) = \begin{bmatrix} x^T \dot{Q}(t)x + f^T(x,t)Q(t)x + x^T Q(t)f(x,t) \\ \dot{h}^T x + h^T(t)f(x,t) \end{bmatrix}.$$

For this purpose, it suffices to use only two functionals from $\mathcal{K}^*$. If $\varphi(y) = y_1$, then, by virtue of (2.3), we obtain the restriction

$$x^T \dot{Q}(t)x + f^T(x,t)Q(t)x + x^T Q(t)f(x,t) \geq 0, \quad x \in \partial\mathcal{I}_t, \quad t \geq 0, \tag{2.9}$$

where $\partial\mathcal{I}_t = \{x \in \mathcal{I}_t \colon x^T Q(t)x = 0\}$. For $\varphi(y) = y_2$, we obtain the inequality

$$h^T(t)f(0,t) \geq 0, \quad t \geq 0. \tag{2.10}$$

Here, we have used the fact that the relations $x^T Q(t)x \geq 0$ and $h^T(t)x = 0$ imply that $x = 0$. Symmetric matrices with indicated inertia possess this property.

Conditions (2.9) and (2.10) guarantee the invariance of the set $\mathcal{I}_t$ in system (2.8). Condition (2.10) is always satisfied for systems with zero equilibrium position, i.e., $f(0,t) \equiv 0$. The differential system (2.5) can serve as an example of these systems. According to (2.4), we have the matrix inequality

$$\dot{Q}(t) + \alpha(x,t)Q(t) + A^T(x,t)Q(t) + Q(t)A(x,t) \geq 0, \quad x \in \partial\mathcal{I}_t, \quad t \geq 0. \tag{2.11}$$

This matrix inequality, together with the given continuous function $\alpha(x,t)$, guarantees the invariance of the set $\mathcal{I}_t$ for system (2.5).

Inequality (2.11) is a generalization of known conditions for the invariance of an ellipsoidal cone for linear systems [3, 4].

Example 2.4. Consider the linear system

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u, \\ \dot{u} &= C(t)x + D(t)u, \end{aligned} \qquad x \in R^n, \quad u \in R^m, \quad t \geq 0, \tag{2.12}$$

where $A(t)$, $B(t)$, $C(t)$, and $D(t)$ are, respectively, $n \times n$, $n \times m$, $m \times n$, and $m \times m$ matrix functions with elements a_{ij}, b_{ij}, c_{ij}, and d_{ij}. Let us find invariance conditions for the set

$$\mathcal{I}_t = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} : \max_k |x_k| \leq \alpha(t) \min_s u_s \right\}, \tag{2.13}$$

where $\alpha(t) > 0$ is a differentiable function. This set is a normal solid cone representable in the form (2.1) with the operator

$$V \colon R^{n+m} \times [t_0, \infty) \to R^{nm+m}, \quad V(x,u,t) = \begin{bmatrix} u_1^2 e - x^2 \\ \vdots \\ u_m^2 e - x^2 \\ u \end{bmatrix},$$

where $e = \alpha^2[1,\ldots,1]^T$ and $x^2 = [x_1^2,\ldots,x_n^2]^T$. The cone of nonnegative vectors R_+^{nm+m} plays the role of the cone $\mathcal{K}$ in Theorem 2.1.

We rewrite condition (2.3) in the form

$$V(x,u,t) \overset{\mathcal{K}}{\geq} 0, \quad u_s = 0 \implies c_s^T x + d_s^T u \geq 0, \tag{2.14}$$

$$V(x,u,t) \overset{\mathcal{K}}{\geq} 0, \quad \alpha^2 u_s^2 = x_k^2 \implies \alpha\dot{\alpha}u_s^2 + \alpha^2 u_s(c_s^T x + d_s^T u) - x_k(a_k^T x + b_k^T u) \geq 0, \tag{2.15}$$

where a_k^T, b_k^T, c_s^T, and d_s^T are the rows of the corresponding matrices, $k = 1, \dots, n$, and $s = 1, \dots, m$. We have $x = 0$ in condition (2.14), and it reduces to the form $d_{sj} \geq 0$, $j \neq s$. In condition (2.15), we have $|x_i| \leq |x_k| = \alpha u_s \leq \alpha u_j \ \ \forall i, j$. If $x_k > 0$, then (2.15) follows from the relations

$$\alpha d_{sj} - b_{kj} \geq 0, \quad j \neq s,$$

$$\dot{\alpha} + \alpha(\alpha c_{sk} - a_{kk}) + \sum_j (\alpha d_{sj} - b_{kj}) \geq \alpha \sum_{i \neq k} |\alpha c_{si} - a_{ki}|.$$

Indeed,

$$\alpha\dot{\alpha}u_s^2 + \alpha^2 u_s(c_s^T x + d_s^T u) - x_k(a_k^T x + b_k^T u) = \alpha u_s w_{sk},$$

$$w_{sk} = [\dot{\alpha} + \alpha(\alpha c_{sk} - a_{kk}) + \alpha d_{ss} - b_{ks}]\, u_s + \sum_{i \neq k} (\alpha c_{si} - a_{ki}) x_i + \sum_{j \neq s} (\alpha d_{sj} - b_{kj}) u_j$$

$$\geq \left[\dot{\alpha} + \alpha(\alpha c_{sk} - a_{kk}) + \alpha d_{ss} - b_{ks} - \alpha \sum_{i \neq k} |\alpha c_{si} - a_{ki}| \right] u_s + \sum_{j \neq s} (\alpha d_{sj} - b_{kj}) u_j$$

$$\geq \left[\dot{\alpha} + \alpha(\alpha c_{sk} - a_{kk}) + \sum_j (\alpha d_{sj} - b_{kj}) - \alpha \sum_{i \neq k} |\alpha c_{si} - a_{ki}| \right] u_s \geq 0.$$

If $x_k < 0$, then, using (2.15), we obtain the following restrictions on the coefficients:

$$\alpha d_{sj} + b_{kj} \geq 0, \quad j \neq s,$$

$$\dot{\alpha} - \alpha(\alpha c_{sk} + a_{kk}) + \sum_j (\alpha d_{sj} + b_{kj}) \geq \alpha \sum_{i \neq k} |\alpha c_{si} + a_{ki}|.$$

Using analogous estimates, we get

$$w_{sk} = [\dot{\alpha} - \alpha(\alpha c_{sk} + a_{kk}) + \alpha d_{ss} + b_{ks}]\, u_s + \sum_{i \neq k} (\alpha c_{si} + a_{ki}) x_i + \sum_{j \neq s} (\alpha d_{sj} + b_{kj}) u_j$$

$$\geq \left[\dot{\alpha} - \alpha(\alpha c_{sk} + a_{kk}) + \alpha d_{ss} + b_{ks} - \alpha \sum_{i \neq k} |\alpha c_{si} + a_{ki}| \right] u_s + \sum_{j \neq s} (\alpha d_{sj} + b_{kj}) u_j$$

$$\geq \left[\dot{\alpha} - \alpha(\alpha c_{sk} + a_{kk}) + \sum_j (\alpha d_{sj} + b_{kj}) - \alpha \sum_{i \neq k} |\alpha c_{si} + a_{ki}| \right] u_s \geq 0.$$

Thus, necessary and sufficient conditions for the positivity of system (2.12) with respect to cone (2.13) have the form

$$\alpha d_{sj} \geq |b_{kj}|, \quad j \neq s,$$

$$\dot{\alpha} \pm \alpha(\alpha c_{sk} \mp a_{kk}) + \sum_{j}(\alpha d_{sj} \mp b_{kj}) \geq \alpha \sum_{i \neq k} |\alpha c_{si} \mp a_{ki}|, \tag{2.16}$$

where $k, i = \overline{1, n}$ and $s, j = \overline{1, m}$. To establish the necessity of these conditions, we set

$$x_k = \pm\alpha\, u_s, \quad x_i = -\mathrm{sign}(\alpha c_{si} \mp a_{ki})\alpha u_s, \quad i \neq k,$$

and consider the following cases:

(1) all components of the vector u coincide;

(2) one component of u is significantly greater than the other components.

Every function $\alpha(t) > 0$ that satisfies the system of inequalities (2.16) is associated with the invariant cone (2.13) of system (2.12).

Note that the system of inequalities (2.16) can be used for the construction of a control in the form of a dynamical compensator that guarantees the positive stabilization of system (2.12).

3. Differential Systems of Higher Order

Consider the following differential system of order $s+1$:

$$X^{(s+1)} = F(X, X^{(1)}, \ldots, X^{(s)}, t), \quad X \in \mathcal{X}, \quad t \geq 0, \tag{3.1}$$

where $F\colon \mathcal{X} \times \ldots \times \mathcal{X} \times [0, \infty) \to \mathcal{X}$ is an operator that satisfies conditions for the existence and uniqueness of a solution $X(t) = X^{(0)}(t)$ with initial conditions $X^{(i)}(t_0) = X_0^{(i)} \in \Omega_i$, $i = 0, \ldots, s$. The complete state of the system is characterized by the functions $X^{(i)}(t)$ that satisfy the first-order differential system

$$\begin{gathered} \dot{X}_0 = X_1, \\ \ldots\ldots\ldots \\ \dot{X}_{s-1} = X_s, \\ \dot{X}_s = F(X_0, \ldots, X_s, t). \end{gathered} \tag{3.2}$$

For this reason, we define invariant sets of system (3.1) in the extended phase space, i.e., in the space of system (3.2), as follows:

$$\mathcal{I}_t = \left\{ (X_0, \ldots, X_s) \in \mathcal{X} \times \ldots \times \mathcal{X}\colon\ V(X_0, \ldots, X_s, t) \overset{\mathcal{K}}{\geq} 0 \right\}, \tag{3.3}$$

where $V\colon \mathcal{X}\times\ldots\times\mathcal{X}\times[0,\infty)\to\mathcal{E}$ is a certain operator and $\overset{\mathcal{K}}{\geq}$ is the inequality generated by a cone or a wedge $\mathcal{K}$ in the space $\mathcal{E}$. The set $\mathcal{I}_t$ is called an invariant set of system (3.1) if its solutions $X(t)$ possess the property

$$(X_0^{(0)},\ldots,X_0^{(s)})\in\mathcal{I}_{t_0}\quad\Longrightarrow\quad(X^{(0)}(t),\ldots,X^{(s)}(t))\in\mathcal{I}_t,\quad t>t_0\geq 0.$$

We assume that the function V is continuous together with its partial derivatives in the domain $\Omega_0\times\ldots\times\Omega_s\times[0,\infty)$ and construct the operator $D_tV(X_0,\ldots,X_s,t)$ of differentiation along the trajectories of system (3.2).

Theorem 3.1. *Let $\mathcal{K}$ be a solid cone. The set $\mathcal{I}_t$ is an invariant set of system (3.1) if and only if the following condition is satisfied for every $t\geq 0$:*

$$(X_0,\ldots,X_s)\in\mathcal{I}_t,\ \varphi\left(V(X_0,\ldots,X_s,t)\right)=0\Longrightarrow\varphi\left(D_tV(X_0,\ldots,X_s,t)\right)\geq 0,\tag{3.4}$$

where $\varphi\in\mathcal{K}^*$.

The proofs of Theorems 3.1 and 2.1 are analogous. Since systems (3.1) and (3.2) are equivalent, Theorem 3.1 can be regarded as a corollary of Theorem 2.1.

Example 3.1. Consider the second-order differential system

$$\ddot{x}+B(t)\dot{x}+A(t)x=0,\tag{3.5}$$

where $A(t)$ and $B(t)$ are bounded matrices. We construct system (3.2) and the function V that describes a set $\mathcal{I}_t$ of the type (3.3) in the form

$$\dot{z}=M(t)z,\quad V(x,y,t)=z^TQ(t)z,$$

where

$$M(t)=\begin{bmatrix}0 & I\\ -A(t) & -B(t)\end{bmatrix},\quad Q(t)=\begin{bmatrix}P(t) & L^T(t)\\ L(t) & R(t)\end{bmatrix},\quad z=\begin{bmatrix}x\\ y\end{bmatrix}.$$

Using the expression

$$D_tV(x,y,t)+\alpha V(x,y,t)=z^T(\dot{Q}+\alpha Q+M^TQ+QM)z=z^THz,$$

we obtain sufficient conditions for the invariance of the set $\mathcal{I}_t$ for system (3.1) in the form of a matrix inequality, namely,

$$H=\begin{bmatrix}\dot{P}+\alpha P-A^TL-L^TA & \dot{L}^T+\alpha L^T-L^TB-A^TR+P\\ \dot{L}+\alpha L-B^TL-RA+P & \dot{R}+\alpha R-B^TR-RB+L+L^T\end{bmatrix}\geq 0.\tag{3.6}$$

Here, for simplicity, we have not taken into account the dependence of all parameters on arguments.

Consider the case of an autonomous system. We set

$$Q = \begin{bmatrix} S + B^T RB & B^T R \\ RB & R \end{bmatrix}, \tag{3.7}$$

where S and R are symmetric matrices. Then inequality (3.6) takes the form

$$H = \begin{bmatrix} \alpha(S + B^T RB) - A^T RB - B^T RA & \alpha B^T R + S - A^T R \\ \alpha RB + S - RA & \alpha R \end{bmatrix} \geq 0.$$

We use a known criterion for the nonnegative definiteness of a block matrix with nonsingular diagonal block, namely,

$$\begin{bmatrix} P & L^T \\ L & R \end{bmatrix} \geq 0 \quad \Longleftrightarrow \quad R > 0, \quad P \geq L^T R^{-1} L.$$

We obtain the following result for the matrix H:

Corollary 3.1. *Suppose that $R = R^T < 0$ and, for a certain $\alpha < 0$, the following matrix inequality is true:*

$$\alpha^2 S - \alpha(B^T S + SB) - (S - A^T R)R^{-1}(S - RA) \leq 0. \tag{3.8}$$

Then the autonomous system (3.5) has the invariant set

$$\mathcal{I} = \{(x, y) \in R^n \times R^n\colon\ x^T(S + B^T RB)x + 2y^T RBx + y^T Ry \geq 0\}. \tag{3.9}$$

Remark 3.1. For $S < 0$, there always exists $\alpha < 0$ such that inequality (3.8) is true. However, in this case, we have $Q < 0$ and $\mathcal{I} = \{0\}$. If $i(S) = \{1, n-1, 0\}$, then $i(Q) = \{1, 2n-1, 0\}$ and set (3.9) is the union of two opposite ellipsoidal cones in the extended phase space of system (3.5). The case where $S = A^T R + RA > 0$ is also of interest. In this case, relation (3.8) is somewhat simplified and, under the conditions of Corollary 3.1, according to the Lyapunov theorem, it is necessary that A and B be Hurwitz matrices.

Corollary 3.2. *If, for $t \geq 0$, a certain function $\alpha(t)$ satisfies the system of inequalities*

$$\begin{gathered} b_{sj}(t) \leq -\frac{1}{\alpha(t)} < 0, \quad j \neq s, \\ \dot{\alpha}(t) - \alpha(t)\sum_j b_{sj}(t) \geq |\alpha^2(t)a_{sk}(t) + 1| + \alpha^2(t)\sum_{i\neq k}|a_{si}(t)|, \end{gathered} \tag{3.10}$$

where $i, j, k, s = \overline{1, n}$, then system (3.5) has the invariant cone

$$\mathcal{I}_t = \left\{(x, y) \in R^n \times R^n\colon\ \max_k |x_k| \leq \alpha(t)\,\min_s y_s\right\}. \tag{3.11}$$

The last statement is a corollary of criterion (2.16) for the positivity of system (2.12) (see, e.g., Example 2.4).

4. Comparison and Ordering of Differential Systems

In the theory of stability of dynamical systems, one uses comparison methods based on the mapping of the space of states of the main system into the space of states of an auxiliary comparison system (see, e.g., [7, 8]). Comparison systems are constructed in the classes of positive or monotone systems with respect to given cones. Time-varying cones in comparison problems were proposed in [5].

We give a general method for the comparison of differential systems that is a corollary of the method for the construction of invariant sets presented in Sec. 2. This method enables one to compare the dynamical properties of two or more dynamical systems acting in different spaces.

Consider a family of independent systems

$$(\mathcal{S}_i):\ \dot{X}_i = F_i(X_i, t), \quad X_i \in \mathcal{X}_i, \quad t \geq 0, \quad i = \overline{1, s}. \tag{4.1}$$

For simplicity, we introduce the notation

$$X = (X_1, \ldots, X_s), \quad F(X, t) = (F_1(X_1, t), \ldots, F_s(X_s, t)), \quad \mathcal{X} = \mathcal{X}_1 \times \ldots \times \mathcal{X}_s.$$

Let $\mathcal{E}$ be a certain space that contains a wedge $\mathcal{K}$ and let an operator $W\colon \mathcal{X} \times [0, \infty) \to \mathcal{E}$ be defined. Assume that every initial condition $X(t_0) = X_0 \in \Omega$ is associated with a unique solution $X(t)$ of the family of systems (4.1) in a certain domain $\Omega \subset \mathcal{X}$ for $t \geq t_0 \geq 0$, and $W(X, t)$ is a continuous function together with its partial derivatives in the domain $\widehat{\Omega} = \Omega \times [0, \infty)$. Furthermore, we assume that the operator W is not everywhere positive with respect to $\mathcal{K}$.

Definition 4.1. *Systems (4.1) are called comparable if, for any $t_0 \geq 0$, the following condition is satisfied:*

$$W(X(t_0), t_0) \overset{\mathcal{K}}{\geq} 0 \implies W(X(t), t) \overset{\mathcal{K}}{\geq} 0, \quad t > t_0. \tag{4.2}$$

In this case, W is the operator of comparison of these systems.

We construct the operator $D_t W(X, t)$ of differentiation along the trajectories of systems (4.1) and formulate the following result:

Theorem 4.1. *Let $\mathcal{K}$ be a solid cone. Then systems (4.1) are comparable if and only if the following condition is satisfied for every $t \geq 0$:*

$$W(X, t) \overset{\mathcal{K}}{\geq} 0, \quad \varphi \in \mathcal{K}^*, \quad \varphi(W(X, t)) = 0 \implies \varphi(D_t W(X, t)) \geq 0. \tag{4.3}$$

The last statement is an obvious corollary of Theorem 2.1.

Let us formulate the main statements of the known comparison principle for two and three systems with zero equilibrium positions, which can be regarded as corollaries of Theorem 4.1.

First, let $s = 2$. We set $W(X, t) = X_2 - V(X_1, t)$, where $V\colon \mathcal{X}_1 \times [0, \infty) \to \mathcal{X}_2$ is an everywhere positive operator with respect to a normal solid cone $\mathcal{K} \subset \mathcal{X}_2$. Then, using the cone inequality

$$D_t V(X_1, t) \overset{\mathcal{K}}{\leq} F_2(V(X_1, t), t) \tag{4.4}$$

and the fact that F_2 belongs to the class of quasimonotone operators $F \in \mathcal{F}$ defined by condition (1.3) with cone $\mathcal{K}_t = \mathcal{K}$, we establish the following property of solutions of the systems:

$$0 \overset{\mathcal{K}}{\leq} V(X_1(t_0), t_0) \overset{\mathcal{K}}{\leq} X_2(t_0) \implies 0 \overset{\mathcal{K}}{\leq} V(X_1(t), t) \overset{\mathcal{K}}{\leq} X_2(t), \quad t > t_0 \geq 0.$$

This means that condition (4.2) is satisfied, i.e., systems (4.1) are comparable in the sense of Definition 4.1.

Assume that the comparison operator V possesses the additional properties

$$V(0, t) \equiv 0, \quad \|V(X, t)\| \geq v(X) > 0, \quad X \neq 0, \quad t \geq 0, \tag{4.5}$$

where $v(x) \geq 0$ is a continuous function such that $v(0) = 0$ and $v(X) \leq v(Y) \implies \|X\| \leq \|Y\|$. Then the following statement is true:

Theorem 4.2. *Suppose that a positive operator V satisfies relations (4.4) and (4.5) and, moreover, $F_2 \in \mathcal{F}$ and $F_1(0, t) \equiv F_2(0, t) \equiv 0$. Then the solution $X_1 \equiv 0$ of system $(\mathcal{S}_1)$ is Lyapunov-stable (asymptotically stable) if the solution $X_2 \equiv 0$ of system $(\mathcal{S}_2)$ is stable (asymptotically stable) in $\mathcal{K}$.*

We now consider the case $s = 3$ and construct a comparison operator in the block form:

$$W(X, t) = [V(X_2, t) - X_1,\ X_3 - V(X_2, t)],$$

where $V\colon \mathcal{X}_2 \times [0, \infty) \to \mathcal{X}_1$ is a certain operator. Assume that the spaces $\mathcal{X}_1$ and $\mathcal{X}_3$ coincide and contain a normal solid cone $\mathcal{K}_1$. In this case, if $F_1 \in \mathcal{F}$, $F_3 \in \mathcal{F}$, and the cone inequalities

$$F_1(V(X_2, t), t) \overset{\mathcal{K}_1}{\leq} D_t V(X_2, t) \overset{\mathcal{K}_1}{\leq} F_3(V(X_2, t), t) \tag{4.6}$$

are true, then the solutions of system $(\mathcal{S}_2)$ can be compared with solutions of systems $(\mathcal{S}_1)$ and $(\mathcal{S}_3)$ as follows:

$$X_1(t_0) \overset{\mathcal{K}_1}{\leq} V(X_2(t_0), t_0) \overset{\mathcal{K}_1}{\leq} X_3(t_0) \implies X_1(t) \overset{\mathcal{K}_1}{\leq} V(X_2(t), t) \overset{\mathcal{K}_1}{\leq} X_3(t), \quad t > t_0 \geq 0.$$

This means that condition (4.2) with cone $\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_1$ is satisfied, i.e., systems (4.1) are comparable according to Definition 4.1. It is easy to see that, in this case, condition (4.3) is a corollary of relations (4.6) and the assumptions concerning F_1 and F_3. Under condition (4.2), systems $(\mathcal{S}_1)$ and $(\mathcal{S}_3)$ are, respectively, the lower comparison system and the upper comparison system for $(\mathcal{S}_2)$ (see, e.g., [5, 8]).

Theorem 4.3. *Suppose that an operator V satisfies relations (4.5) and (4.6) and, furthermore, $F_1 \in \mathcal{F}$, $F_3 \in \mathcal{F}$, and $F_i(0, t) \equiv 0$, $i = \overline{1, 3}$. Then the solution $X_2 \equiv 0$ of system $(\mathcal{S}_2)$ is Lyapunov-stable (asymptotically stable) if the solution $X_1 \equiv 0$ of system $(\mathcal{S}_1)$ is stable (asymptotically stable) in $-\mathcal{K}_1$ and the solution $X_3 \equiv 0$ of system $(\mathcal{S}_3)$ is stable (asymptotically stable) in $\mathcal{K}_1$.*

Note that Theorems 4.2 and 4.3 are also true under the conditions $F_2 \in \overline{\mathcal{F}}_2$, $F_1 \in \underline{\mathcal{F}}_1$, and $F_3 \in \overline{\mathcal{F}}_1$, where $\overline{\mathcal{F}}_2$, $\underline{\mathcal{F}}_1$, and $\overline{\mathcal{F}}_1$ are certain more general classes of operators defined with the use of the varying normal reproducing cone $\mathcal{K}_t$ [5].

For a family of $s \geq 2$ independent systems, the problems of *ordering* and finding a *dominating* (in a certain sense) system are formulated in the form of a general comparison problem. Indeed, consider the block operator

$$W(X,t) = \left[\; V_2(X_2,t) - V_1(X_1,t), \;\; \ldots \;\; , V_s(X_s,t) - V_{s-1}(X_{s-1},t) \;\right], \tag{4.7}$$

where $V_i\colon \mathcal{X}_i \times [0,\infty) \to \mathcal{E}_1$, $i = \overline{1,s}$, are certain operators. Assume that the space $\mathcal{E}_1$ contains a wedge $\mathcal{K}_1$ and condition (4.2), where $\mathcal{K} = \mathcal{K}_1 \times \ldots \times \mathcal{K}_1$, is satisfied. Then the solutions of the family of systems (4.1) are ordered as follows:

$$V_1(X_1(t),t) \overset{\mathcal{K}_1}{\leq} V_2(X_2(t),t) \overset{\mathcal{K}_1}{\leq} \ldots \overset{\mathcal{K}_1}{\leq} V_s(X_s(t),t), \quad t > t_0, \tag{4.8}$$

provided that this ordering takes place at an arbitrary initial time $t = t_0 \geq 0$. In particular, if $V_i(X_i,t) = \|X_i\|_{\mathcal{X}_i}$ is the norm in the space $\mathcal{X}_i$, then the norms of solutions of systems (4.1) are ordered as follows:

$$\|X_1(t_0)\|_{\mathcal{X}_1} \leq \ldots \leq \|X_s(t_0)\|_{\mathcal{X}_s} \quad \Longrightarrow \quad \|X_1(t)\|_{\mathcal{X}_1} \leq \ldots \leq \|X_s(t)\|_{\mathcal{X}_s}, \quad t > t_0.$$

For the identical operators $V_i = E$, we have

$$X_1(t_0) \overset{\mathcal{K}_1}{\leq} \ldots \overset{\mathcal{K}_1}{\leq} X_s(t_0) \quad \Longrightarrow \quad X_1(t) \overset{\mathcal{K}_1}{\leq} \ldots \overset{\mathcal{K}_1}{\leq} X_s(t), \quad t > t_0.$$

Moreover, system $(\mathcal{S}_s)$ is dominating in the family of systems (4.1).

In the case of the solid cone $\mathcal{K}$, Theorem 4.1 gives a criterion for this kind of ordering of the family of systems (4.1) in the form (4.3).

Example 4.1. Consider a family of systems

$$\dot{X}_i = A_i(X_i,t)X_i, \quad X_i \in R^{n_i}, \quad t \geq 0, \quad i = \overline{1,s}, \tag{4.9}$$

where A_i are $n_i \times n_i$ matrices that depend continuously on X_i and t.

We specify operator (4.7) by setting

$$V_i(X_i,t) = X_i^T Q_i(t) X_i, \quad Q_i(t) \equiv Q_i^T(t), \quad i = \overline{1,s}.$$

Then

$$\lambda_{\min}(H_i) X_i^T X_i \leq D_t V_i(X_i,t) = X_i^T H_i X_i \leq \lambda_{\max}(H_i) X_i^T X_i,$$

where $H_i = A_i^T Q_i + Q_i A_i + \dot{Q}_i$. Using Theorem 4.1, one can establish that, for the ordering of systems (4.9) in the form (4.8) with cone $\mathcal{K} = R_+^{s-1}$, it is sufficient that the following relations hold in the domain $\widehat{\Omega}$:

$$H_j \leq \beta_j Q_j, \quad \alpha_{j+1} Q_{j+1} \leq H_{j+1}, \quad \beta_j \leq \alpha_{j+1}, \quad j = \overline{1, s-1}, \tag{4.10}$$

where $\beta_j(X_j,t)$ and $\alpha_{j+1}(X_{j+1},t)$ are certain continuous scalar functions. If all matrices $Q_i > 0$ are positive definite, then the following estimates are satisfied in (4.10):

$$\beta_j \geq \lambda_{\max}(H_j - \lambda Q_j), \quad \alpha_{j+1} \leq \lambda_{\min}(H_{j+1} - \lambda Q_{j+1}), \quad j = \overline{1, s-1},$$

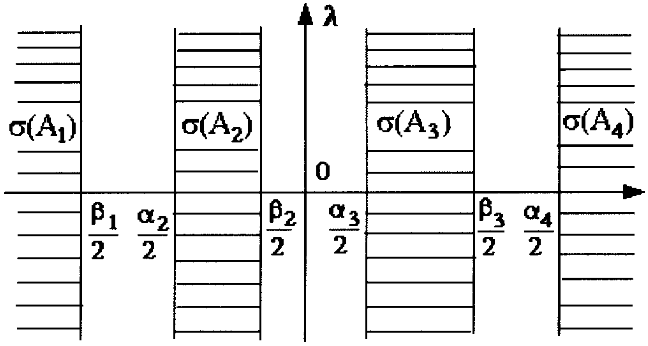

Fig. 1. Domains of possible location of the spectra $\sigma(A_i)$ under the ordering conditions (4.10) for $s = 4$ systems.

where $\lambda_{\max}(\cdot)$ $(\lambda_{\min}(\cdot))$ is the maximum (minimum) eigenvalue of the corresponding pencil of matrices. In this case, we have the following sufficient conditions for the ordering of systems (4.9) in the form (4.8):

$$\lambda_{\max}(H_j - \lambda Q_j) \le \lambda_{\min}(H_{j+1} - \lambda Q_{j+1}), \quad j = \overline{1, s-1}.$$

Let all matrices Q_i be time-independent and positive definite. Then, in the case where the matrix inequalities in (4.10) are true, the spectra of the matrices A_i must be located in the corresponding domains; furthermore, the neighboring domains can have common points only on boundary lines (Fig. 1).

If $Q_i \equiv I$ and, in the domain $\widehat{\Omega}$, the inequalities

$$\lambda_{\max}(A_j^T + A_j) \le \lambda_{\min}(A_{j+1}^T + A_{j+1}), \quad j = \overline{1, s-1},$$

are true, then the solutions of systems (4.9) are ordered with respect to the Euclidean norm, i.e.,

$$\|X_1(t_0)\| \le \ldots \le \|X_s(t_0)\| \quad \Longrightarrow \quad \|X_1(t)\| \le \ldots \le \|X_s(t)\|, \quad t > t_0.$$

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