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ON FUNDAMENTAL SOLUTIONS OF THE CAUCHY PROBLEM FOR A CLASS OF DEGENERATE PARABOLIC EQUATIONS

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We present the results of an investigation and some applications of fundamental solutions of the Cauchy problem for a new class of parabolic equations. In these equations: (i) there exist three groups of spatial variables, one basic and two auxiliary, (ii) different weights of spatial variables from the basic group with respect to the time variable are admitted, (iii) degeneracies in variables from the auxiliary groups are present, (iv) a degeneracy on the initial hyperplane is present.

In the theory of the Cauchy problem for parabolic equations (and systems of equations), one of the most important notions is the notion of fundamental solution. At present, the most precise and complete results for the fundamental solution of the Cauchy problem are obtained in the case of equations uniformly parabolic by Petrovsky (where all the spatial variables have the same status and the same weight $2b$ with respect to the time variable). These results were generalized to the following cases: Eidelman $\overrightarrow{2b}$-parabolic equations, where each spatial variable can have its own vector parabolic weight $\overrightarrow{2b}=(2b_1,\dots,2b_n)$ [7, 11, 13], degenerate parabolic equations which generalize the Kolmogorov classical equation of diffusion with inertia [6, 8–10, 12, 16], equations parabolic by Petrovsky, $\overrightarrow{2b}$-parabolic equations, and degenerate equations of the Kolmogorov type which have certain degeneracies on the initial hypersurface [1–4].

Recently, S. D. Eidelman and one of the authors [5, 15] defined and started the investigation of a new class of parabolic equations, namely, degenerate equations of the Kolmogorov type with $\overrightarrow{2b}$-parabolic part in the main group of variables. In these equations, the definitions of $\overrightarrow{2b}$-parabolicity and the structure of equations of the Kolmogorov type were generalized. Moreover, these equations can also be pseudodifferential.

In this paper, we consider equations from this new class in the case where the coefficients of equations do not depend on the spatial variables and degeneracies are present on the initial hypersurface. We construct the fundamental solution of the Cauchy problem, investigate its properties, and present theorems on well-defined solvability of the Cauchy problem and on the integral representation of solutions for homogeneous equations with weak degeneracy on the initial hypersurface.

1. We use the following notation: $n_1, n_2, n_3, b_1,\dots,b_{n_1}$ are given natural numbers and, furthermore, $n_1\ge n_2\ge n_3$, $N\equiv n_1+n_2+n_3$, $\overrightarrow{2b}\equiv(2b_1,\dots,2b_{n_1})$, $q_j\equiv 2b_j/(2b_j-1)$, $1\le j\le n_1$, $\mathbb{Z}_+^r$ is the set of all r-dimensional multiindices,

$$\|m_1\|\equiv\sum_{j=1}^{n_1}\frac{m_{1j}}{2b_j}\quad\text{if}\quad m_1\equiv(m_{1j},1\le j\le n_1)\in\mathbb{Z}_+^{n_1},$$

$$M_m\equiv\sum_{l=1}^{3}\sum_{j=1}^{n_l}\left(l-1+\frac{1}{2b_j}\right)(m_{lj}+1)\quad\text{if}\quad m\equiv(m_{lj},1\le j\le n_l,\ 1\le l\le 3)\in\mathbb{Z}_+^N,$$

$$\{X\equiv(x_1,x_2,x_3),\ \Xi\equiv(\xi_1,\xi_2,\xi_3\}\subset\mathbb{R}^N\quad\text{if}\quad\{x_l\equiv(x_{lj},1\le j\le n_l),\ \xi_l\equiv(\xi_{lj},1\le j\le n_l)\}\subset\mathbb{R}^{n_l},\quad 1\le l\le 3,$$

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$$\partial_{x_1}^{m_1} \equiv \prod_{j=1}^{n_1} \partial_{x_{1j}}^{m_{1j}}, \quad \partial_X^m \equiv \prod_{l=1}^{3} \prod_{j=1}^{n_l} \partial_{x_{lj}}^{m_l} \quad \text{if} \quad x_1 \in \mathbb{R}^{n_1}, \quad X \in \mathbb{R}^N, \quad m_1 \in \mathbb{Z}_+^{n_1}, \quad m \in \mathbb{Z}_+^N,$$

$$B(t, \tau) \equiv \int_\tau^t \frac{\beta(\gamma)}{\alpha(\gamma)} d\gamma, \quad X_{lj}(t, \tau) \equiv \sum_{r=1}^{l-1} \frac{1}{r!} (B(t, \tau))^r x_{(l-r)j}, \quad 1 \le j \le n_l, \quad 1 \le l \le 3,$$

$$X(t, \tau) \equiv (X_{lj}(t, \tau),\ 1 \le j \le n_l,\ 1 \le l \le 3),$$

$$E_c(t, X; \tau, \Xi) \equiv \exp\left\{ -c \sum_{l=1}^{3} \sum_{j=1}^{n_l} (B(t, \tau))^{1 - lq_j} \left| X_{lj}(t, \tau) - \xi_{lj} \right|^{q_j} \right\},$$

$$E_c^d(t, X; \tau, \Xi) \equiv E_c(t, X; \tau, \Xi) \exp\left\{ d \int_\tau^t \frac{d\gamma}{\alpha(\gamma)} \right\}, \quad \Pi_H \equiv \left\{ (t, X) \mid t \in H,\ X \in \mathbb{R}^N \right\},$$

T is a given positive number, and i is the imaginary unit.

Consider an equation of the form

$$(Lu)(t, X) \equiv \left(\alpha(t)\partial_t - \beta(t) \left(\sum_{l=2}^{3} \sum_{j=1}^{n_l} x_{(l-1)j} \partial_{x_{lj}} + \sum_{0 < \|m_1\| \le 1} a_{m_1}(t) \partial_{x_1}^{m_1} \right) - a_0(t) \right) u(t, X) = 0, \quad (t, X) \in \Pi_{(0, T]}, \tag{1}$$

where the functions α, β: $[0, T] \to \mathbb{R}$, $a_{m_1}: [0, T] \to \mathbb{C}$, $0 < \|m_1\| \le 1$, a_0: $(0, T] \to \mathbb{C}$, are continuous and such that $\alpha(0)\beta(0) = 0$, $\forall t \in (0, T]$: $\alpha(t) > 0$, $\beta(t) > 0$, where β is monotonically nondecreasing, the differential expression $\partial_t - \sum_{\|m_1\| \le 1} a_{m_1}(t) \partial_{x_1}^{m_1}$, $t \in [0, T]$, is $\overrightarrow{2b}$-parabolic [11, 13], and $\exists A \in \mathbb{R}$ $\forall t \in (0, T]$: $\mathrm{Re}\, a_0(t) \le A$.

2. The fundamental solution of the Cauchy problem for Eq. (1) is defined as the function $Z(t, X; \tau, \Xi)$, $0 < \tau < t \le T$, $\{X, \Xi\} \subset \mathbb{R}^N$, such that the function

$$u(t, X) \equiv \int_{\mathbb{R}^N} Z(t, X; \tau, \Xi) \varphi(\Xi) d\Xi, \quad (t, X) \in \Pi_{(\tau, T]}, \tag{2}$$

is a solution of Eq. (1) which satisfies the condition

$$u(t, X)\big|_{t=\tau} = \varphi(X), \quad X \in \mathbb{R}^N, \tag{3}$$

for any number $\tau \in (0, T)$ and for an arbitrary continuous bounded function φ: $\mathbb{R}^N \to \mathbb{C}$.

One of the basic results of the present paper is the following theorem:

Theorem 1. *The following statements are true:*

(i) there exists the unique fundamental solution $Z(t, X; \tau, \Xi)$, $0 < \tau < t \le T$, $\{X, \Xi\} \subset \mathbb{R}^N$, *of the Cauchy problem for Eq. (1),*