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GENERALIZATION OF CONTINUED FRACTIONS. I

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We constructed a new algebraic object, namely, recursion fractions of the nth order that are n-dimensional generalizations of continued fractions. For the representation and the study of such fractions, we used paradeterminants and triangular matrices.

At present, various approaches to the generalization of calculation of rational approximations of continued fractions are known [1–3, 6]. Among them, the historically first approach is the matrix one (Euler, Jacobi, Poincaré, Brunn, Perron, Bernstein, and Pustyl'nikov). Another approach is based on linear homogeneous forms (Dirichlet, Hermite, Klein, Minkowski, Voronoi, Skubenko, and Arnol'd). A number of algorithms were proposed by the analysts Hurwitz and Sekeresh (on the basis of generalizations of Farey fractions), Skorobagat'ko (branching continued fractions), Syavavko (integral continued fractions), etc.

An important generalization of continued fractions was proposed by Fürshtenau [7] as early as 1874. Later on, some sufficient conditions of convergence of rational approximations of Fürshtenau fractions were studied by Krukovs'kyi [5]. However, despite the natural character and simplicity of this generalization, it remained unnoticed or was forgotten.

The important requirements to a generalization of continued fractions are as follows:

– construction of an algebraic object convenient in use, whose representation recalls the representation of continued fractions and allows one to naturally introduce the notion of their order and to separate a class of periodic objects as generalizations of the periodic continued fractions;

– the algorithm of calculation of the values of rational truncations of these mathematical objects must be simple to realize and efficient;

– by analogy with periodic continued fractions, arbitrary periodic algebraic objects of higher orders must serve as representations of some algebraic irrationalities of higher orders.

The representation used by Fürshtenau in the generalization of fractions is not convenient for the operation with such fractions. We propose a new representation for Fürshtenau fractions, by using the parapermanents of triangular matrices. It is more descriptive and allows one to include the calculation apparatus for triangular matrices to the study of these fractions [4]. In addition, this approach allows us to naturally introduce the notion of order of a fraction and a periodic fraction and to show that periodic fractions of higher orders represent irrationalities of higher orders.

1. Auxiliary Notions and Assertions

Recall some information from [4] on the parapermanents of triangular matrices. Let a field K be given.

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Definition 1. *A triangular table of elements from a field K of the form*

$$A = \left\| \begin{matrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \cdots & \cdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{matrix} \right\|_n \tag{1}$$

is called a triangular matrix, and the number n is its order.

In correspondence to each element a_{ij} of matrix (1), we put $i-j+1$ elements a_{ik}, $k=j,\ldots,i$, called derivative elements of the matrix that are generated by the key element a_{ij}. We denote the product of all derivative elements generated by the element a_{ij} by $\{a_{ij}\}$ and call a factorial product of the key element a_{ij}, i.e.,

$$\{a_{ij}\} = \prod_{k=j}^{i} a_{ik}.$$

Definition 2. *A collection of elements of matrix (1) is called the normal collection of key elements of this matrix if they generate the set of derivative elements of power n, such that each two elements do not belong to the same column of this matrix.*

By $\mathcal{P}(n)$, we denote the set of all ordered partitions of a natural number n into a sum of natural terms. It is known that $|\mathcal{P}(n)| = 2^{n-1}$. There exists the bijective correspondence between the normal collections of key elements of matrix (1) and the ordered partitions from the set $\mathcal{P}(n)$:

$$(n_1, n_2, \ldots, n_r) \leftrightarrow (a_{N_1, N_0+1}, a_{N_2, N_1+1}, \ldots, a_{N_r, N_{r-1}+1}),$$

where

$$N_0 = 0, \qquad N_s = \sum_{i=1}^{s} n_i, \qquad s = 1, 2, \ldots, r.$$

Definition 3. *The parapermanent of the triangular matrix (1) is the element*

$$\operatorname{pper}(A) = \begin{bmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \cdots & \cdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_n = \sum_{(\alpha_1, \alpha_2, \ldots, \alpha_r) \in \mathcal{P}(n)} \prod_{s=1}^{r} \{a_{i(s), j(s)}\}$$

of the field K, where $a_{i(s),j(s)}$ is a key element corresponding to the sth component of a partition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$.

Definition 4. *In correspondence to each element a_{ij} of the triangular matrix A given by formula (1), we put the triangular matrix*

$$R_{ij}(A) = \left\| \begin{matrix} a_{jj} & & & \\ a_{j+1,j} & a_{j+1,j+1} & & \\ \dots & \dots & \ddots & \\ a_{ij} & a_{i,j+1} & \dots & a_{ii} \end{matrix} \right\|_{i-j+1} .$$

We call it an a_{ij}*-part of the given triangular matrix.*

We assume that

$$\operatorname{pper}(R_{01}(A)) = \operatorname{pper}(R_{n,n+1}(A)) = 1 .$$

The parapermanents can be expanded in elements of the first column or the last row of a triangular matrix, respectively, by the formulas

$$\operatorname{pper}(A) = \sum_{r=1}^{n} \{a_{r1}\} \cdot \operatorname{pper}(R_{n,r+1}) = \sum_{s=1}^{n} \{a_{ns}\} \cdot \operatorname{pper}(R_{s-1,1}) .$$

2. Definition of Recursion Fractions

Definition 5. *A recursion fraction of the* n*th order is the triangular matrix*

$$\alpha = \left[\begin{array}{c|ccccccc} a_{11} & & & & & & & \\ \frac{a_{22}}{a_{12}} & a_{12} & & & & & & \\ \dots & \dots & \ddots & & & & & \\ \frac{a_{n-1,n-1}}{a_{n-2,n-1}} & \frac{a_{n-2,n-1}}{a_{n-3,n-1}} & \dots & a_{1,n-1} & & & & \\ \frac{a_{n,n}}{a_{n-1,n}} & \frac{a_{n-1,n}}{a_{n-2,n}} & \dots & \frac{a_{2,n}}{a_{1,n}} & a_{1,n} & & & \\ 0 & \frac{a_{n,n+1}}{a_{n-1,n+1}} & \dots & \frac{a_{3,n+1}}{a_{2,n+1}} & \frac{a_{2,n+1}}{a_{1,n+1}} & a_{1,n+1} & & \\ 0 & 0 & \dots & \frac{a_{4,n+2}}{a_{3,n+2}} & \frac{a_{3,n+2}}{a_{2,n+2}} & \frac{a_{2,n+2}}{a_{1,n+2}} & a_{1,n+2} & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \ddots \end{array} \right]_{\infty} , \tag{2}$$

where a_{ij} *are natural numbers.*

Definition 6. *The* m*th rational truncation of a fraction* α *is the rational number*

$$\alpha_m = \frac{P_m}{Q_m} , \tag{3}$$

where P_m, Q_m *are the parapermanents of relevant triangular matrices:*

$$P_m = \begin{bmatrix} a_{11} & & & & & & & \\ \frac{a_{22}}{a_{12}} & a_{12} & & & & & & \\ \dots & \dots & \ddots & & & & & \\ \frac{a_{n,n}}{a_{n-1,n}} & \frac{a_{n-1,n}}{a_{n-2,n}} & \dots & a_{1,n} & & & & \\ 0 & \frac{a_{n,n+1}}{a_{n-1,n+1}} & \dots & \frac{a_{2,n+1}}{a_{1,n+1}} & a_{1,n+1} & & & \\ 0 & 0 & \dots & \frac{a_{3,n+2}}{a_{2,n+2}} & \frac{a_{2,n+2}}{a_{1,n+2}} & a_{1,n+2} & & \\ \dots & \dots & \dots & \dots & \dots & \dots & \ddots & \\ 0 & 0 & \dots & 0 & \frac{a_{n,m}}{a_{n-1,m}} & \frac{a_{n-1,m}}{a_{n-2,m}} & \dots & a_{1,m} \end{bmatrix}_m, \tag{4}$$

$$Q_m = \begin{bmatrix} a_{12} & & & & & & & \\ \frac{a_{23}}{a_{13}} & a_{13} & & & & & & \\ \dots & \dots & \ddots & & & & & \\ \frac{a_{n-1,n}}{a_{n-2,n}} & \frac{a_{n-2,n}}{a_{n-3,n}} & \dots & a_{1,n} & & & & \\ \frac{a_{n,n+1}}{a_{n-1,n+1}} & \frac{a_{n-1,n+1}}{a_{n-2,n+1}} & \dots & \frac{a_{2,n+1}}{a_{1,n+1}} & a_{1,n+1} & & & \\ 0 & \frac{a_{n,n+2}}{a_{n-1,n+2}} & \dots & \frac{a_{3,n+2}}{a_{2,n+2}} & \frac{a_{2,n+2}}{a_{1,n+2}} & a_{1,n+2} & & \\ \dots & \dots & \dots & \dots & \dots & \dots & \ddots & \\ 0 & 0 & \dots & 0 & \frac{a_{n,m}}{a_{n-1,m}} & \frac{a_{n-1,m}}{a_{n-2,m}} & \dots & a_{1,m} \end{bmatrix}_{m-1}. \tag{5}$$

We denote the ratio of these parapermanents by

$$\frac{P_m}{Q_m} = \left[\begin{array}{c|ccccccc} a_{11} & & & & & & & \\ \frac{a_{22}}{a_{12}} & a_{12} & & & & & & \\ \dots & \dots & \ddots & & & & & \\ \frac{a_{n,n}}{a_{n-1,n}} & \frac{a_{n-1,n}}{a_{n-2,n}} & \dots & a_{1,n} & & & & \\ 0 & \frac{a_{n,n+1}}{a_{n-1,n+1}} & \dots & \frac{a_{2,n+1}}{a_{1,n+1}} & a_{1,n+1} & & & \\ 0 & 0 & \dots & \frac{a_{3,n+2}}{a_{2,n+2}} & \frac{a_{2,n+2}}{a_{1,n+2}} & a_{1,n+1} & & \\ \dots & \dots & \dots & \dots & \dots & \dots & \ddots & \\ 0 & 0 & \dots & 0 & \frac{a_{n,m}}{a_{n-1,m}} & \frac{a_{n-1,m}}{a_{n-2,m}} & \dots & a_{1,m} \end{array}\right]_m.$$

If the limit

$$\lim_{m\to\infty} \frac{P_m}{Q_m}$$

exists, we call it the value of the recursion fraction α. Expanding parapermanents (4) and (5) in elements of the last row, we obtain linear recursion relations of the nth order,

$$P_m = a_{1m}P_{m-1} + a_{2m}P_{m-2} + \ldots + a_{nm}P_{m-n}, \qquad m = 1,2,\ldots,$$

$$Q_m = a_{1m}Q_{m-1} + a_{2m}Q_{m-2} + \ldots + a_{nm}Q_{m-n}, \qquad m = 1,2,\ldots,$$

where

$$P_i = \begin{cases} 1, & i = 0, \\ 0, & i < 0, \end{cases} \qquad Q_i = \begin{cases} 1, & i = 1-n, \\ 0, & 2-n \le i \le 0, \end{cases} \qquad a_{n1} = 1,$$

which present an efficient algorithm of calculation of the rational truncations (3) of the n-order recursion fractions (2). This implies that a recursion fraction of the second order for $a_{1i} = q_i > 0$, $a_{2i} = p_i$ has the form

$$\left[\begin{array}{c|cccccc} q_1 & & & & & & \\ \frac{p_2}{q_2} & q_2 & & & & & \\ 0 & \frac{p_3}{q_3} & q_3 & & & & \\ 0 & 0 & \frac{p_4}{q_4} & q_4 & & & \\ \ldots & \ldots & \ldots & \ldots & \ddots & & \\ 0 & 0 & 0 & 0 & \ldots & q_m & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ddots \end{array}\right]_\infty$$

and is another representation of the continued fraction [8, 9]

$$q_1 + \mathop{\mathrm{K}}_{m=2}^{\infty} \frac{p_m}{q_m}. \tag{6}$$

Definition 7. *The nth order recursion fraction (2) is called k-periodic if $a_{i,rk+j} = a_{i,j}$, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,k$.*

Definition 8. *Two nth order recursion fractions are called identical if the mth rational truncations of both fractions are identical for all $m = 1,2,\ldots$.*

Definition 9. *The nth order recursion fraction (2) is called an ordinary recursion fraction if*

$$a_{nn} = a_{n,n+1} = a_{n,n+2} = \ldots = 1.$$

3. Properties of Recursion Fractions

Let us set the recursion fraction

$$\alpha = \left[\begin{array}{c|cccccccccc}
q_1 &&&&&&&&&& \\
\frac{p_2}{q_2} & q_2 &&&&&&&&& \\
\frac{r_3}{p_3} & \frac{p_3}{q_3} & q_3 &&&&&&&& \\
0 & \frac{r_4}{p_4} & \frac{p_4}{q_4} & q_4 &&&&&&& \\
\dots & \dots & \dots & \dots & \ddots &&&&&& \\
0 & 0 & 0 & 0 & \dots & q_{n-2} &&&&& \\
0 & 0 & 0 & 0 & \dots & \frac{p_{n-1}}{q_{n-1}} & q_{n-1} &&&& \\
0 & 0 & 0 & 0 & \dots & \frac{s\cdot r_n}{s\cdot p_n} & \frac{s\cdot p_n}{s\cdot q_n} & s\cdot q_n &&& \\
0 & 0 & 0 & 0 & \dots & 0 & \frac{s\cdot r_{n+1}}{s\cdot p_{n+1}} & \frac{s\cdot p_{n+1}}{q_{n+1}} & q_{n+1} && \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & \frac{s\cdot r_{n+2}}{p_{n+2}} & \frac{p_{n+2}}{q_{n+2}} & q_{n+2} & \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \ddots
\end{array}\right]_{\infty}.$$

Theorem 1. *For any recursion fraction of the kth order, there exists an ordinary recursion fraction of the kth order that is equal to it.*

Proof of this theorem will be illustrated for $k = 3$ and $k = 4$. (In the general case, the proof is analogous.) First of all, we note that the value of the parameter s does not affect the value of a recursion fraction of the third order, because this recursion fraction is considered as the formal ratio of the parapermanents of two infinite triangular matrices, in which s is cancelled. Hence, the value of the parameter s can be chosen according to the described procedure so that the equality $sr_n = 1$ is satisfied. Moreover, by performing successively this procedure for elements $r_3, r_4, \dots, r_k$ of a recursion fraction, we determine the values of the corresponding parameters $s_3, s_4, \dots, s_k$. To this end, it is sufficient to successively solve the equations

$$s_3 r_3 = s_3 s_4 r_4 = s_3 s_4 s_5 r_5 = s_4 s_5 s_6 r_6 = s_5 s_6 s_7 r_7 = \dots = s_{k-2} s_{k-1} s_k r_k = 1$$

for the variables $s_3, s_4, \dots s_k$:

$$s_{3t} = \frac{r_5 r_8 \cdot \ldots \cdot r_{3t-1}}{r_3 r_6 \cdot \ldots \cdot r_{3t}}, \qquad s_{3t+1} = \frac{r_3 r_6 \cdot \ldots \cdot r_{3t}}{r_4 r_7 \cdot \ldots \cdot r_{3t+1}}, \qquad s_{3t+2} = \frac{r_4 r_7 \cdot \ldots \cdot r_{3t-2}}{r_5 r_8 \cdot \ldots \cdot r_{3t-1}}, \qquad t = 1,2,\dots.$$

By reducing the recursion fractions of the fourth order to ordinary fourth-order recursion fractions, it is necessary to use the equalities

$$t_{4k} = \frac{s_7 \cdot \ldots \cdot s_{4k-1}}{s_4 s_8 \cdot \ldots \cdot s_{4k}}, \qquad t_{4k+1} = \frac{s_4 s_8 \cdot \ldots \cdot s_{4k}}{s_5 s_9 \cdot \ldots \cdot s_{4k+1}},$$

$$t_{4k+2} = \frac{s_5 s_9 \cdot \ldots \cdot s_{4k+1}}{s_6 s_{10} \cdot \ldots \cdot s_{4k+2}}, \qquad t_{4k+3} = \frac{s_6 s_{10} \cdot \ldots \cdot s_{4k+2}}{s_7 \cdot \ldots \cdot s_{4k+3}}, \qquad k = 1,2,\ldots .$$

Theorem 2. *If there exists a finite nonzero limit for the mth rational truncation of the second-order 1-periodic recursion fraction*

$$\left[\begin{array}{c|ccccc} a_1 & & & & & \\ \frac{a_2}{a_1} & a_1 & & & & \\ 0 & \frac{a_2}{a_1} & a_1 & & & \\ \ldots & \ldots & \ldots & \ddots & & \\ 0 & 0 & \ldots & \frac{a_2}{a_1} & a_1 & \\ 0 & 0 & \ldots & 0 & \frac{a_2}{a_1} & a_1 \end{array}\right]_m \tag{7}$$

as $m \to \infty$, then its value is a real root of the quadratic equation $x^2 = a_1 x + a_2$.

Proof. The assertion of this theorem follows directly from the fact that the continued fractions (6) for $q_1 = q_2 = \ldots = a_1$, $p_2 = p_3 = \ldots = a_2$ are another representation of the recursion fraction (7).

A 1-periodic recursion fraction of the third order takes the form

$$\left[\begin{array}{c|ccccc} a_1 & & & & & \\ \frac{a_2}{a_1} & a_1 & & & & \\ \frac{a_3}{a_2} & \frac{a_2}{a_1} & a_1 & & & \\ 0 & \frac{a_3}{a_2} & \frac{a_2}{a_1} & a_1 & & \\ 0 & 0 & \frac{a_3}{a_2} & \frac{a_2}{a_1} & a_1 & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ddots \end{array}\right]_\infty . \tag{8}$$

Theorem 3. *If there exists a finite nonzero limit for the mth rational truncation of the third-order 1-periodic recursion fraction (8) as $m \to \infty$, then such third-order recursion fraction is a representation of a real root of the cubic equation*

$$x^3 = a_1 x^2 + a_2 x + a_3. \tag{9}$$

Proof. By expanding the parapermanent which is the numerator of the mth rational truncation in elements of the first column, we obtain the equality

$$P_m = a_1 P_{m-1} + a_2 P_{m-2} + a_3 P_{m-3}.$$

Since $Q_m = P_{m-1}$, $m = 1,2,\dots$, we have

$$\frac{P_m}{Q_m} = a_1 + \frac{a_2}{\frac{P_{m-1}}{Q_{m-1}}} + \frac{a_3}{\frac{P_{m-1}}{Q_{m-1}} \frac{P_{m-2}}{Q_{m-2}}}. \tag{10}$$

Let

$$\lim_{m\to\infty} \frac{P_m}{Q_m} = x \neq 0.$$

Then, by passing to the limit as $m \to \infty$ in equality (10), we obtain an equation equivalent to Eq. (9):

$$x = a_1 + \frac{a_2}{x} + \frac{a_3}{x^2}. \tag{11}$$

We note that Eq. (11) can be written in the form

$$x = a_1 + \frac{a_2 + \frac{a_3}{x}}{x}.$$

Using the relation

$$x_n = a_1 + \frac{a_2 + \frac{a_3}{x_{n+1}}}{x_{n+1}}, \qquad n = 1,2,\dots,$$

and the corresponding successive substitutions, we obtain the expression

$$x = x_1 = a_1 + \cfrac{a_2 + \cfrac{a_3}{a_1 + \cfrac{a_2 + \cfrac{a_3}{a_1 + \dots}}{a_1 + \cfrac{a_2 + \dots}{a_1 + \dots}}}}{a_1 + \cfrac{a_2 + \cfrac{a_3}{a_1 + \cfrac{a_2 + \dots}{a_1 + \dots}}}{a_1 + \cfrac{a_2 + \cfrac{a_3}{a_1 + \dots}}{a_1 + \cfrac{a_2 + \dots}{a_1 + \dots}}}}. \tag{12}$$

The same multilevel fractions can be constructed with the help of two sequences of equalities,

$$\left\{x_n = a_1 + \frac{y_n}{x_{n+1}}\right\}_{n=1,2,\ldots}, \qquad \left\{y_n = a_2 + \frac{a_3}{x_{n+1}}\right\}_{n=1,2,\ldots},$$

and the corresponding substitutions [5]. Formulas (12) are not convenient for their analysis and practical needs. Therefore, we will use the corresponding third-order recursion fractions equal to them.

4. 2-Periodic and 3-Periodic Third-Order Recursion Fractions

Let us consider the 2-periodic third-order recursion fraction

$$\left[\begin{array}{c|ccccc} q_1 & & & & & \\ \frac{p_2}{q_2} & q_2 & & & & \\ \frac{r_1}{p_1} & \frac{p_1}{q_1} & q_1 & & & \\ 0 & \frac{r_2}{p_2} & \frac{p_2}{q_2} & q_2 & & \\ 0 & 0 & \frac{r_1}{p_1} & \frac{p_1}{q_1} & q_1 & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ddots \end{array}\right]_\infty . \tag{13}$$

We now expand the numerator of the nth rational truncation $\frac{P_n}{Q_n}$ of this recursion fraction in elements of the first column:

$$\frac{P_n}{Q_n} = \frac{q_1 Q_n + p_2 P_{n-2} + r_1 Q_{n-2}}{Q_n} = q_1 + \frac{p_2}{\frac{Q_n}{P_{n-2}}} + \frac{r_1}{\frac{Q_n}{P_{n-2}}\frac{P_{n-2}}{Q_{n-2}}}.$$

Expanding the denominator Q_n of this rational truncation in elements of the first column, we obtain the equalities

$$\frac{Q_n}{P_{n-2}} = \frac{q_2 P_{n-2} + p_1 Q_{n-2} + r_2 P_{n-4}}{P_{n-2}} = q_2 + \frac{p_1}{\frac{P_{n-2}}{Q_{n-2}}} + \frac{r_2}{\frac{P_{n-2}}{Q_{n-2}}\frac{Q_{n-2}}{P_{n-4}}}.$$

Assume that the following finite nonzero limits exist:

$$\lim_{n\to\infty} \frac{P_n}{Q_n} = x \quad \text{and} \quad \lim_{n\to\infty} \frac{Q_n}{P_{n-2}} = y.$$

Then

$$x = q_1 + \frac{p_2}{y} + \frac{r_1}{yx}, \qquad y = q_2 + \frac{p_1}{x} + \frac{r_2}{xy}.$$

This system of equations yields the cubic equation

$$(q_2 p_2 + r_2)x^3 + (r_1 q_2 - p_2 q_1 q_2 - p_2^2 + p_1 p_2 - 2q_1 r_2)x^2 + (p_1 r_1 - p_1 p_2 q_1$$
$$- q_1 q_2 r_1 - 2 p_2 r_1 + q_1^2 r_2)x - (r_1^2 + q_1 p_1 r_1) = 0. \tag{14}$$

Theorem 4. *Let* $\frac{P_n}{Q_n}$ *be the* n*th rational truncation of the periodic recursion equation (13), and let the nonzero limits*

$$\lim_{n\to\infty} \frac{P_n}{Q_n} = x \qquad \text{and} \qquad \lim_{n\to\infty} \frac{Q_n}{P_{n-2}} = y$$

exist. Then x *is a real root of the cubic equation (14).*

Analogously, we can show that the 3-periodic third-order recursion fractions represent the real root of some cubic equation, etc.

Example. In the 2-periodic third-order recursion fraction (13), we take

$$q_1 = 5, \quad q_2 = -3, \quad p_1 = 1, \quad p_2 = -2, \quad r_1 = 3, \quad r_2 = -1.$$

Then

$$\frac{P_0}{Q_0} = \frac{5}{1} = 5, \quad \frac{Q_2}{P_0} = \frac{-14}{5} = -2.8, \quad \frac{P_1}{Q_1} = \frac{-17}{-3} = 5.(6), \quad \frac{Q_3}{P_1} = \frac{47}{-17} \approx -2.764,$$

$$\frac{P_2}{Q_2} = \frac{-77}{-14} = 5.5, \quad \frac{Q_4}{P_2} = \frac{212}{-77} \approx -2.7532, \quad \frac{P_3}{Q_3} = \frac{260}{47} \approx 5.5319,$$

$$\frac{Q_5}{P_3} = \frac{-716}{260} \approx -2.753846, \quad \frac{P_4}{Q_4} = \frac{1172}{212} \approx 5.5283, \quad \frac{Q_6}{P_4} = \frac{-3227}{1172} \approx -2.7534129,$$

$$\frac{P_5}{Q_5} = \frac{-3959}{-716} \approx 5.529329, \quad \frac{Q_7}{P_5} = \frac{10901}{-3959} \approx -2.75347309,$$

$$\frac{P_6}{Q_6} = \frac{-17843}{-3227} \approx 5.529284, \quad \frac{Q_8}{P_6} = \frac{49130}{-17843} \approx -2.75346076,$$

$$\frac{P_7}{Q_7} = \frac{60275}{10901} \approx 5.52930923, \qquad \frac{Q_9}{P_7} = \frac{-165965}{60275} \approx -2.75346329,$$

$$\frac{P_8}{Q_8} = \frac{271655}{49130} \approx 5.52930999, \qquad \frac{Q_{10}}{P_8} = \frac{-747992}{271655} \approx -2.753463032,$$

$$\frac{P_9}{Q_9} = \frac{-917672}{-165965} \approx 5.529310396.$$

The fraction is a representation of the root

$$x = \frac{7}{3} + \frac{1}{3}\sqrt[3]{55} + \frac{2}{15}\sqrt[3]{55^2} \approx 5.52931047609$$

of the cubic equation

$$5x^3 - 35x^2 + 45x - 24 = 0.$$

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