INVARIANT CONES AND STABILITY OF LINEAR DYNAMICAL SYSTEMS

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UDC 517.983.27

We present a method for the investigation of the stability and positivity of systems of linear differential equations of arbitrary order. Conditions for the invariance of classes of cones of circular and ellipsoidal types are established. We propose algebraic conditions for the exponential stability of linear positive systems based on the notion of maximal eigenpairs of a matrix polynomial.

Introduction

In the modeling of complex technical, biological, and other objects, one uses differential or difference systems of equations in the phase spaces of which invariant sets (in particular, cones) exist. These specific features of systems should be taken into account and used in qualitative research methods and in problems of the analysis of stability and control (see, e.g., [1–3]).

In the present paper, we propose a method for the investigation of the positivity and stability of linear dynamical systems in a semiordered space. For the analysis of stability of these systems, we develop special methods based on the spectral properties of positive and positive-invertible operators. We establish conditions for the invariance of circular-type cones and their generalizations, which enable one, in particular, to solve the problem of the positive stabilization of systems with respect to these cones by using dynamic compensators. Conditions for the invariance of ellipsoidal cones and the exponential stability of linear differential and difference systems are formulated in the form of matrix inequalities. Using the notion of maximal eigenpairs of a matrix polynomial, we propose algebraic conditions for the exponential stability of linear differential systems of arbitrary order.

1. Definitions and Auxiliary Facts

For a symmetric matrix $S = S^T \in R^{n\times n}$, the triple of numbers $i(S) = \{ i_+(S), i_-(S), i_0(S) \}$, where $i_+(S)$, $i_-(S)$, and $i_0(S)$ are the numbers of, respectively, positive, negative, and zero eigenvalues of S (counting multiplicities) is called the inertia of S.

We give some definitions and facts from the theory of cones and operators in a semiordered space. A convex closed set $\mathcal{K}$ of a real normed space $\mathcal{E}$ is called a wedge if $\alpha\mathcal{K} + \beta\mathcal{K} \subset \mathcal{K}$ for any $\alpha, \beta \geq 0$. A wedge $\mathcal{K}$ with edge $\mathcal{K} \cap -\mathcal{K} = \{0\}$ is a cone. The conjugate cone $\mathcal{K}^*$ is formed by linear functionals $\varphi \in \mathcal{E}$ that take nonnegative values on elements of $\mathcal{K}$, and, furthermore, $\mathcal{K} = \left\{ X \in \mathcal{E} : \varphi(X) \geq 0 \;\; \forall \varphi \in \mathcal{K}^* \right\}$. The space with a cone is semiordered: $X \leq Y \Leftrightarrow Y - X \in \mathcal{K}$. A cone $\mathcal{K}$ with a nonempty set of interior points $\operatorname{int}\mathcal{K} = \{ X : X > 0 \}$ is a solid cone. A cone $\mathcal{K}$ is called normal if the relation $0 \leq X \leq Y$ implies that $\|X\| \leq \nu \|Y\|$, where ν is a universal constant. The least number ν of this sort is the constant of normality of the cone. If $\mathcal{E} = \mathcal{K} - \mathcal{K}$,

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then the cone $\mathcal{K}$ is reproducing. A cone $\mathcal{K}$ is normal only if the conjugate cone $\mathcal{K}^*$ is reproducing. The set of vectors with nonnegative elements and the set of symmetric nonnegative-definite matrices can serve as examples of normal reproducing cones in finite-dimensional spaces.

Let a cone $\mathcal{K}_1$ ($\mathcal{K}_2$) be selected in a Banach space $\mathcal{E}_1$ ($\mathcal{E}_2$). An operator $M: \mathcal{E}_1 \to \mathcal{E}_2$ is called monotone if the relation $X \geq Y$ implies that $MX \geq MY$. The monotonicity of a linear operator is equivalent to its positivity: $X \geq 0 \Rightarrow MX \geq 0$. If $M\mathcal{E}_1 \subset \mathcal{K}_2$, then the operator M is everywhere positive. A linear operator M is called positive invertible if $\mathcal{K}_2 \subset M\mathcal{K}_1$, i.e., for any $Y \in \mathcal{K}_2$, the equation $MX = Y$ has a solution $X \in \mathcal{K}_1$. If $\mathcal{K}_2$ is a normal reproducing cone and $M_1 \leq M \leq M_2$, then the positive invertibility of the operators M_1 and M_2 yields the positive invertibility of the operator M, and, furthermore, $M_2^{-1} \leq M^{-1} \leq M_1^{-1}$ [1]. A criterion for the positive invertibility of a class of operators $M = L - P$, $P\mathcal{K}_1 \subset \mathcal{K}_2 \subset L\mathcal{K}_1$, where $\mathcal{K}_2$ is a normal reproducing cone, is the inequality $\rho(T) < 1$ ($\rho(T)$ is the spectral radius of the operator pencil $T(\lambda) = P - \lambda L$) [4]. If $\mathcal{K}_2$ is a solid cone, then this inequality is equivalent to the condition $M\mathcal{K}_1 \cap \operatorname{int}\mathcal{K}_2 \neq \varnothing$.

Let $X(t) = \Phi(t, t_0, X_0) \in \mathcal{E}$ be a state of a certain dynamical system described by a continuously differentiable function for $t \geq t_0 \geq 0$. If an operator $\Omega(t, t_0): \mathcal{E} \to \mathcal{E}$ that uniquely determines the transition from the initial state $X(t_0) = X_0$ to the state $X(t)$ for $t > t_0$ is given, then $\Phi(t, t_0, X_0) = \Omega(t, t_0)X_0$. Moreover,

$$\Omega(t_0, t_0) = E, \quad \Omega(t + \tau, t_0) = \Omega(t, \tau) \cdot \Omega(\tau, t_0) \quad \forall\, t, \tau \geq t_0,$$

where E is the identity operator. The system has an invariant set $\mathcal{K}_t \subset \mathcal{E}$ if, for any $t_0 \geq 0$, it follows from the inclusion $X_0 \in \mathcal{K}_0$ that $X(t) \in \mathcal{K}_t$ for $t \geq t_0$. If $\mathcal{K}_t$ is a cone, then inequalities between elements of the space generated by this cone at every moment of time t are denoted by $\overset{\mathcal{K}_t}{\leq}$ and $\overset{\mathcal{K}_t}{\geq}$.

Let us define properties of systems with respect to a variable cone [5]. A dynamical system possessing an invariant cone $\mathcal{K}_t$ is positive with respect to this cone. A system is called monotone with respect to the cone $\mathcal{K}_t$ if, for any $t_0 \geq 0$, one has

$$X_{10} \overset{\mathcal{K}_0}{\leq} X_{20} \;\Rightarrow\; X_1(t) \overset{\mathcal{K}_t}{\leq} X_2(t), \quad t > t_0, \tag{1.1}$$

where $X_k(t) = \Phi(t, t_0, X_{k0})$, $k = 1, 2$. Denote the classes of positive and monotone systems by $\mathcal{M}_0$ and $\mathcal{M}$. The classes of systems possessing property (1.1) under the additional restrictions $X_{20} \in \mathcal{K}_0$, $X_{10} \in \mathcal{K}_0$, $X_{10} \in -\mathcal{K}_0$, and $X_{20} \in -\mathcal{K}_0$ are denoted by $\mathcal{M}_1^+$, $\mathcal{M}_2^+$, $\mathcal{M}_1^-$, and $\mathcal{M}_2^-$, respectively. Using elements of the conjugate cone, we can establish that the differential system

$$\dot{X} = F(X, t), \quad X \in \mathcal{E}, \quad t \geq 0, \tag{1.2}$$

belongs to the indicated classes. In particular, system (1.2) is positive and monotone with respect to the solid cone $\mathcal{K}_t$ if $t < \tau \Rightarrow \mathcal{K}_t \subseteq \mathcal{K}_\tau$ and the following conditions are satisfied:

$$X \overset{\mathcal{K}_t}{\geq} 0, \quad \varphi \in \mathcal{K}_t^*, \quad \varphi(X) = 0 \;\Rightarrow\; \varphi(F(X, t)) \geq 0, \tag{1.3}$$

$$X \overset{\mathcal{K}_t}{\leq} Y, \quad \varphi \in \mathcal{K}_t^*, \qquad \varphi(X - Y) = 0 \;\Rightarrow\; \varphi\big(F(Y, t) - F(X, t)\big) \geq 0, \tag{1.4}$$

where $\mathcal{K}_t^*$, $t \geq 0$, is the conjugate cone.

We say that an isolated equilibrium state $X \equiv 0$ of a dynamical system is stable in $\mathcal{K}_t$ if, for any $\varepsilon > 0$ and $t_0 \geq 0$, one can indicate $\delta > 0$ such that the condition $X_0 \in \mathcal{S}_\delta(t_0)$ implies that $X(t) \in \mathcal{S}_\varepsilon(t)$ for $t > t_0$, where $\mathcal{S}_\varepsilon(t) = \{X \in \mathcal{K}_t : \|X\| \leq \varepsilon\}$. If, in addition, it follows from the inclusion $X_0 \in \mathcal{S}_{\delta_0}(t_0)$ for a certain $\delta_0 > 0$ that $\|X(t)\| \to 0$ as $t \to \infty$, then the state $X \equiv 0$ of the system is asymptotically stable in $\mathcal{K}_t$. If the state $X \equiv 0$ of a system with invariant cone $\mathcal{K}_t$ is Lyapunov stable (asymptotically stable), then it is stable (asymptotically stable) in $\mathcal{K}_t$.

For dynamical systems with discrete time, the invariant sets and the properties of positivity and monotonicity with respect to a cone and stability in $\mathcal{K}_t$ are defined by analogy.

2. Cones of Circular and Ellipsoidal Types

In the space R^{n+1}, we consider the set

$$\mathcal{K}(Q, h) = \left\{ z \in R^{n+1} : z^T Q z \geq 0,\; z^T Q h \geq 0 \right\}, \tag{2.1}$$

where $Q = Q^T$ is a symmetric matrix with inertia $i(Q) = \{1, n, 0\}$ and h is an arbitrary vector such that $h^T Q h > 0$. The hyperplane $\mathcal{P} = \left\{ z : z^T Q h = 0 \right\}$ separates the sets $\mathcal{K}(Q, h)$ and $-\mathcal{K}(Q, h)$ and passes through their unique common point $z = 0$. It is obvious that $\mathcal{K}(Q, h) = \mathcal{K}(Q, h_1)$ for any interior vector $h_1 \in$ int $\mathcal{K}(Q, h)$. In particular, h may be the eigenvector of the matrix Q corresponding to its unique positive eigenvalue [6].

Lemma 2.1. *The set $\mathcal{K}(Q, h)$ is a cone.*

Proof. It is known that $i_+(Q) = 1$ if and only if [7]

$$S = Q - \frac{1}{\omega} Q h h^T Q \leq 0,$$

where $\omega = h^T Q h > 0$. If $z_1 \in \mathcal{K}(Q, h)$ and $z_2 \in \mathcal{K}(Q, h)$, then, using the factorization $S = -R^T R$ and the Cauchy inequality, we obtain

$$\frac{1}{\omega} z_1^T Q h h^T Q z_1 + z_1^T S z_1 = \alpha^2 - a^T a \geq 0,$$

$$\alpha = \frac{1}{\sqrt{\omega}} z_1^T Q h \geq 0, \qquad a = R z_1,$$

$$\frac{1}{\omega} z_2^T Q h h^T Q z_2 + z_2^T S z_2 = \beta^2 - b^T b \geq 0,$$

$$\beta = \frac{1}{\sqrt{\omega}} z_2^T Q h \geq 0, \quad b = R z_2,$$

$$\frac{1}{\omega} z^T Q h h^T Q z + z^T S z = \alpha^2 - a^T a + \beta^2 - b^T b + 2(\alpha\beta - a^T b) \geq 0,$$

where $z = z_1 + z_2$. Therefore, $z_1 + z_2 \in \mathcal{K}(Q, h)$.

If $z \in \pm\mathcal{K}(Q, h)$, then $z^T Q h = 0$, $z^T Q z = z^T S z = 0$, $Qz = Sz = 0$, and $z = 0$. Here, we have taken into account the nonsingularity of Q and the equivalence of the relations $z^T S z = 0$ and $Sz = 0$ for a matrix $S \leq 0$.

The property of the cone $\alpha\mathcal{K}(Q, h) \subset \mathcal{K}(Q, h)$ for $\alpha \geq 0$ is obvious.

The lemma is proved.

The set of interior points of the cone $\mathcal{K}(Q, h)$, its boundary, and the conjugate cone, respectively, have the following forms:

$$\operatorname{int}\mathcal{K}(Q, h) = \left\{ z \in \mathcal{K}(Q, h) \colon z^T Q z > 0,\ z^T Q h > 0 \right\},$$

$$\partial\mathcal{K}(Q, h) = \left\{ z \in \mathcal{K}(Q, h) \colon z^T Q z = 0 \right\}, \qquad \mathcal{K}^*(Q, h) = Q\mathcal{K}(Q, h).$$

Let T be the nonsingular matrix of the transformation

$$T^T Q T = \Delta \overset{\Delta}{=} \operatorname{diag}\{-1, \ldots, -1, 1\}, \qquad h = Tg.$$

Then $\mathcal{K}(Q, h) = T\mathcal{K}(\Delta, g)$, and, furthermore, $\mathcal{K}(\Delta, e)$, $e = [0, \ldots, 0, 1]^T$, coincides with the Minkowski circular cone

$$\mathcal{K}(\Delta) = \left\{ z \in R^{n+1} \colon z^T = \left[x^T, u \right],\ \|x\| \leq u \right\}, \tag{2.2}$$

where $\|x\| = \sqrt{x^T x}$. Therefore, $\mathcal{K}(Q, h) = \alpha T\mathcal{K}(\Delta)$, where $\alpha = e^T T^{-1} h$, i.e., $\mathcal{K}(Q, h)$ coincides with $T\mathcal{K}(\Delta)$ $(-T\mathcal{K}(\Delta))$ if $\alpha > 0$ $(\alpha < 0)$.

Since $\mathcal{K}(\Delta)$ is a normal cone with constant of normality 1, the cone $\mathcal{K}(Q, h)$ is also normal and its constant of normality does not exceed $\sqrt{t_-/t_+}$, where $t_-(t_+)$ is the minimum (maximum) eigenvalue of the matrix TT^T.

We construct the matrix T using the spectral decomposition

$$Q = \gamma h h^T - H\Gamma H^T = GDG^T, \qquad \sigma(Q) = \{-\gamma_1, \ldots, -\gamma_n, \gamma\}, \tag{2.3}$$

where $\gamma > 0$, $\Gamma = \operatorname{diag}\{\gamma_1, \ldots, \gamma_n\} > 0$, $D = \operatorname{diag}\{-\gamma_1, \ldots, -\gamma_n, \gamma\}$, $G = [H, h]$, $h^T h = 1$, $H^T H = I$, $h^T H = 0$, and $GG^T = G^T G = I$. We define cone (2.1) as follows:

$$\mathcal{K}(Q) = \left\{ z \in R^{n+1} \colon z^T Q z \geq 0,\ z^T h \geq 0 \right\}, \tag{2.4}$$

where h is the normalized eigenvector of the matrix Q corresponding to its unique positive eigenvalue γ. In this case, the following relations hold:

$$\mathcal{K}^*(Q) = \mathcal{K}(Q^{-1}) = Q\mathcal{K}(Q), \quad \mathcal{K}(Q) = G\mathcal{K}(D) = T\mathcal{K}(\Delta),$$

$$\mathcal{K}(D) = L\mathcal{K}(\Delta), \quad T = GL, \quad L = \operatorname{diag}\left\{\gamma_1^{-1/2},\ldots,\gamma_n^{-1/2},\gamma^{-1/2}\right\}.$$

Note that the fact that a vector z belongs to the cone $\mathcal{K}(Q)$ (in particular, to $\mathcal{K}(\Delta)$) is described in terms of nonnegative-definite matrices:

$$z \in \mathcal{K}(Q) \;\Leftrightarrow\; u_z \geq 0, \quad \gamma u_z^2\Gamma^{-1} \geq U_z U_z^T \;\Leftrightarrow\; \begin{bmatrix} u_z\Gamma^{-1} & U_z \\ U_z^T & u_z\gamma \end{bmatrix} \geq 0,$$

where $u_z = h^T z$ and $U_z = H^T z$.

The so-called light cone [8]

$$\mathcal{K}_a = \left\{z \in R^{n+1}:\ \|z\| \leq (a, z)\right\}, \tag{2.5}$$

where $(a, z) = a^T z$ is the scalar product and a is a given vector with norm $\|a\| > 1$, belongs to the class of cones of the type $\mathcal{K}(Q)$. Indeed, set (2.5) can be described in the form (2.4) if we set $Q = aa^T - I$ and $h = \|a\|^{-1}a$. In this case, we have $\gamma = a^T a - 1$ and $i(Q) = \{1, n, 0\}$. Since $Q^{-1} = \frac{1}{\gamma}aa^T - I$, we get $\mathcal{K}_a^* = \mathcal{K}_b$, where $b = \frac{1}{\sqrt{\gamma}}a$. For $\|a\| = \sqrt{2}$, the cone $\mathcal{K}_a$ is self-conjugate.

In the space R^{n+m}, we consider the sets of vectors [9]

$$\mathcal{K}_p(\mu_\alpha) = \left\{z \in R^{n+m}: z^T = \left[x^T, u^T\right],\ u \in R_+^m,\ \|x\|_p \leq \mu_\alpha(u)\right\}, \tag{2.6}$$

$$\mathcal{K}_q(\sigma_\beta) = \left\{w \in R^{n+m}: w^T = \left[y^T, v^T\right],\ v \in R_+^m,\ \|y\|_q \leq \sigma_\beta(v)\right\}, \tag{2.7}$$

where $\mu_\alpha(u) = \alpha \min_k u_k$, $\sigma_\beta(v) = \beta\sum_k v_k$, $R_+^m \subset R^m$ is the cone of vectors with nonnegative elements, and $\|a\|_p$ is one of the following vector norms:

$$\|x\|_1 = \sum_k |x_k|, \quad \|x\|_p = \left(\sum_k |x_k|^p\right)^{1/p}, \quad \|x\|_\infty = \max_k |x_k|.$$

Assume that the parameters α, β, p, and q satisfy the relations

$$\alpha\beta = 1, \quad \alpha > 0, \quad \beta > 0, \quad p^{-1} + q^{-1} = 1, \quad p \geq 1, \quad q \geq 1. \tag{2.8}$$

Then, for each of the norms introduced, sets (2.6) and (2.7) are solid cones, and, furthermore, the following inequalities hold:

$$|y^T x| \le \|x\|_p \|y\|_q, \qquad v^T u \ge \mu_\alpha(u)\sigma_\beta(v). \tag{2.9}$$

For $p > 1$ $(q > 1)$, the first inequality in (2.9) is the Hölder inequality.

Lemma 2.2. *Under conditions (2.8), one has* $\mathcal{K}_p^*(\mu_\alpha) = \mathcal{K}_q(\sigma_\beta)$.

Proof. If $z \in \mathcal{K}_p(\mu_\alpha)$ and $w \in \mathcal{K}_q(\sigma_\beta)$, then, according to (2.8) and (2.9), we have

$$y^T x + v^T u \ge -\|x\|_p \|y\|_q + \mu_\alpha(u)\sigma_\beta(v) \ge 0.$$

This means that $\mathcal{K}_q(\sigma_\beta) \subset \mathcal{K}_p^*(\mu_\alpha)$.

The reverse inclusion $\mathcal{K}_q(\sigma_\beta) \supset \mathcal{K}_p^*(\mu_\alpha)$ is also true. Indeed, let $y^T x + v^T u \ge 0$ for an arbitrary vector $z \in \mathcal{K}_p(\mu_\alpha)$. Then we obviously have $v \in R_+^m$. To establish the inequality $\|y\|_q \le \sigma_\beta(v)$, the following cases should be considered:

$$(1)\quad p = 1, \qquad q = \infty, \qquad x_k = \begin{cases} -y_s, & k = s, \\ 0, & k \ne s, \end{cases} \qquad u = \beta\|x\|_1 e;$$

$$(2)\quad p = \infty, \qquad q = 1, \qquad x_k = \begin{cases} -1, & y_k \ge 0, \\ 1, & y_k < 0, \end{cases} \qquad u = \beta\|x\|_\infty e;$$

$$(3)\quad p > 1, \qquad q > 1, \qquad x_k = \begin{cases} -|y_k|^{q/p}, & y_k \ge 0, \\ |y_k|^{q/p}, & y_k < 0, \end{cases} \qquad u = \beta\|x\|_p e.$$

Here, $|y_s| = \|y\|_\infty$ and $e = [1, \dots, 1]^T$. For each of these cases, we have

$$y^T x + v^T u = -\|x\|_p \|y\|_q + \|x\|_p \sigma_\beta(v) \ge 0,$$

whence $\|y\|_q \le \sigma_\beta(v)$, i.e., $w \in \mathcal{K}_q(\sigma_\beta)$.

The lemma is proved.

3. Conditions for Positivity and Stability of Linear Systems

A linear differential system in a Banach space

$$\dot{z} = Mz, \qquad z \in \mathcal{E}, \quad t \ge 0, \tag{3.1}$$

where $M : \mathcal{E} \to \mathcal{E}$ is a bounded operator, has an invariant cone $\mathcal{K}$, i.e., is positive with respect to $\mathcal{K}$, if $e^{Mt}\mathcal{K} \subset \mathcal{K}$ for any $t \ge 0$. A linear difference system

$$z_{k+1} = Mz_k, \quad z_k \in \mathcal{E}, \quad k = 0, 1, \ldots, \tag{3.2}$$

has an invariant cone $\mathcal{K}$ if the operator $M \overset{\mathcal{K}}{\ge} 0$ is positive with respect to $\mathcal{K}$.

Conditions for the existence of invariant and solid cones in spaces of finite-dimensional systems (3.1) and (3.2) are described by using the spectrum $\sigma(M)$ [10, 11]. System (3.1) is positive with respect to a certain solid cone if and only if the following conditions are satisfied:

$$\alpha(M) \overset{\Delta}{=} \max\{\operatorname{Re}\lambda : \lambda \in \sigma(M)\} \in \sigma(M),$$

$$\lambda \in \sigma(M), \quad \operatorname{Re}\lambda = \alpha(M) \Rightarrow d(\lambda) \le d(\alpha(M)),$$

where $d(\cdot)$ is the multiplicity of an eigenvalue of the matrix as a root of its minimal polynomial. Similarly, system (3.2) is positive with respect to a certain solid cone if and only if

$$\rho(M) \overset{\Delta}{=} \max\{|\lambda| : \lambda \in \sigma(M)\} \in \sigma(M),$$

$$\lambda \in \sigma(M), \quad |\lambda| = \rho(M) \Rightarrow d(\lambda) \le d(\rho(M)).$$

Supplementing the last conditions with the inequality $d(\rho(M)) \le 3$ $(d(\rho(M)) \le 2$ if $\rho(M) = 0)$ and requiring that the Jordan canonical form of the matrix M have at most one block of order ≥ 2 with eigenvalues $\lambda \in \sigma(M)$ for $|\lambda| = \rho(M)$, we obtain a criterion for the existence of the invariant ellipsoidal cone (2.4) for system (3.2) [12].

Conditions for the stability of systems (3.1) and (3.2) positive with respect to normal reproducing cones are described in terms of positive solutions of algebraic equations (see, e.g., [13, 14]). In particular, the inclusion $\mathcal{K} \subset (E - M)\mathcal{K}$ is a criterion for the asymptotic stability of the positive system (3.2). For system (3.1), the following statement is true [15]:

Theorem 3.1. *The positive system (3.1) is exponentially stable if and only if the operator* $-M$ *is positive invertible, i.e.,* $\mathcal{K} \subset -M\mathcal{K}$. *If* $\mathcal{K} \subset (\gamma E - M)\mathcal{K}$ $\forall \gamma \ge 0$, *then system (3.1) is exponentially stable and positive with respect to* $\mathcal{K}$.

We establish sufficient conditions for the exponential stability of system (3.1) in the form of the positive invertibility of two operators.

Theorem 3.2. *If, for a certain* γ_0, *one has*

$$\mathcal{K} \subset -M\mathcal{K} \cap (\gamma_0 E - M)\mathcal{K}, \quad \gamma_0 > \frac{\rho^2(M) - r^2(M)}{2r(M)}, \tag{3.3}$$

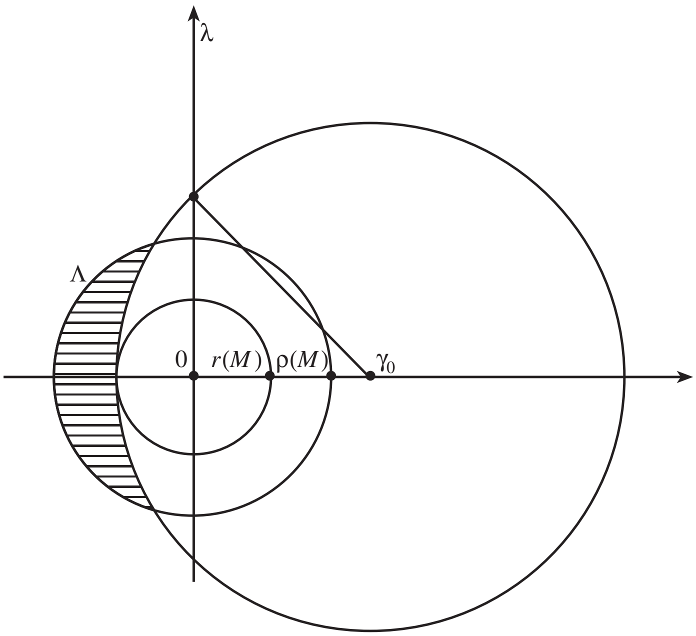

Fig. 1. Λ is the domain of location of the spectrum $\sigma(M)$.

where $\rho(M)$ *is the spectral radius of the operator* M *and* $r(M) \overset{\Delta}{=} \min\{|\lambda|: \lambda \in \sigma(M)\}$, *then system (3.1) is exponentially stable.*

Proof. It follows from (3.3) that the operators $-M^{-1}$ and $(\gamma_0 E - M)^{-1}$ have the invariant cone $\mathcal{K}$. Their spectra consist, respectively, of the numbers $-1/\lambda$ and $1/(\gamma_0 - \lambda)$ for $\lambda \in \sigma(M)$. According to the theorem on the spectral radius of a positive operator, we have

$$|\lambda| \geq -\alpha, \quad |\gamma_0 - \lambda| \geq \gamma_0 - \beta, \quad \lambda \in \sigma(M),$$

where $\alpha, \beta \in \sigma(M)$ are certain real points of the spectrum. If $0 \leq \gamma \leq \gamma_0$, then $-M \overset{\mathcal{K}}{\leq} \gamma E - M \overset{\mathcal{K}}{\leq} \gamma_0 E - M$ and every operator $\gamma E - M$ must be positive invertible (see the theorem on a two-sided estimate for a positive-invertible operator [1]). Thus, in the case considered, α and β coincide and are equal to the number $-r(M)$.

If the estimate for γ_0 in (3.3) is true, then the spectrum of the operator M belongs to a certain domain Λ located to the left of the imaginary axis (see Fig. 1). This is a criterion for the exponential stability of system (3.1).

The theorem is proved.

We establish conditions for the positivity of systems (3.1) and (3.2) with respect to ellipsoidal cones $\mathcal{K}(Q)$ of the type (2.4) and apply them to the problem of analysis of stability.

Lemma 3.1. *If* $P = P^T$, *then* $z^T P z \geq 0 \ \ \forall z \in \mathcal{K}(Q) \Leftrightarrow \exists \alpha \geq 0 : P \geq \alpha Q$.

Proof. It is known that $w^T P w \geq 0$ for $w \in \mathcal{K}(\Delta)$ if and only if there is $\alpha \geq 0$ such that $P \geq \alpha \Delta$ [16].

Taking into account that $\mathcal{K}(Q) = T\mathcal{K}(\Delta)$, setting $z = Tw$, and using the law of inertia, we obtain a criterion for the nonnegativity of the quadratic form $z^T Pz$ on the cone $\mathcal{K}(Q)$ in the form of the matrix inequality $P \geq \alpha Q$.

The lemma is proved.

Denote

$$M = [R, l] = \begin{bmatrix} A & b \\ c^T & d \end{bmatrix}, \quad R = [r_1, \dots, r_n] = \begin{bmatrix} A \\ c^T \end{bmatrix}, \quad l = \begin{bmatrix} b \\ d \end{bmatrix},$$

$$A = [a_1, \dots, a_n], \quad b^T = [b_1, \dots, b_n], \quad c^T = [c_1, \dots, c_n].$$

Theorem 3.3. *The quantity $\mathcal{K}(Q)$ is an invariant cone of system (3.2) if and only if the following conditions are satisfied:*

$$M^T QM \geq \alpha Q, \quad h^T Mh \geq 0, \quad h^T MQ^{-1} M^T h \geq 0, \tag{3.4}$$

where $\alpha \geq 0$ is a certain nonnegative number.

Proof. First, we show that $\mathcal{K}(\Delta)$ is an invariant cone of the matrix M if and only if the following conditions are satisfied:

$$l \in \mathcal{K}(\Delta), \quad M\Delta M^T \geq \alpha\Delta, \tag{3.5}$$

where $\alpha \geq 0$ is a certain nonnegative number. The inclusion $M\mathcal{K}(\Delta) \subset \mathcal{K}(\Delta)$ means that

$$S_z = \begin{bmatrix} uI & x \\ x^T & u \end{bmatrix} \geq 0 \;\Rightarrow\; S_{Mz} = \begin{bmatrix} (c^T x + du)I & Ax + bu \\ x^T A^T + ub^T & c^T x + du \end{bmatrix} \geq 0,$$

i.e., the following relations must hold for arbitrary $z \in \mathcal{K}(\Delta)$ and $g \in R^{n+1}$:

$$g^T S_{Mz} g = l_g^T z \geq 0, \quad l_g^T = \left[g^T S_{r_1} g, \dots, g^T S_{r_n} g, g^T S_l g \right],$$

$$S_l = \begin{bmatrix} dI & b \\ b^T & d \end{bmatrix}, \quad S_{r_i} = \begin{bmatrix} c_i I & a_i \\ a_i^T & c_i \end{bmatrix}, \quad i = 1, \dots, n.$$

Taking into account that the cone $\mathcal{K}(\Delta)$ is self-conjugate, we conclude that $l_g \in \mathcal{K}(\Delta)$, i.e.,

$$g^T S_l g = w^T l \geq 0, \quad (g^T S_l g)^2 - \sum_{i=1}^{n} \left(g^T S_{r_i} g\right)^2 = w^T Sw \geq 0,$$

where

$$g = \begin{bmatrix} y \\ \\ v \end{bmatrix}, \quad w = \Phi(g) = \begin{bmatrix} 2vy \\ \\ y^T y + v^2 \end{bmatrix}, \quad S = ll^T - RR^T = M\Delta M^T.$$

It is easy to establish that the nonlinear transformation $\Phi : R^{n+1} \to R^{n+1}$ preserves the cone $\mathcal{K}(\Delta)$ and, moreover, $\Phi(\mathcal{K}(\Delta)) = \mathcal{K}(\Delta)$. Therefore, we can use Lemma 3.1.

Thus, a criterion for the invariance of the cone $\mathcal{K}(\Delta)$ for the matrix M has the form (3.5).

Since $\mathcal{K}(Q) = T\mathcal{K}(\Delta)$, the conditions $M\mathcal{K}(Q) \subset \mathcal{K}(Q)$ and $M_T\mathcal{K}(\Delta) \subset \mathcal{K}(\Delta)$, where $M_T = T^{-1}MT$, are equivalent. According to decomposition (2.3), the last column of the matrix M_T has the form $l_T = \gamma^{-1/2}T^{-1}Mh$. Hence, conditions (3.5) for the vector l_T and matrix M_T reduce to the form

$$h^T Mh \geq 0, \quad h^T M^T QMh \geq 0, \quad MQ^{-1}M^T \geq \alpha Q^{-1}. \tag{3.6}$$

It is known that the matrix M has the invariant cone $\mathcal{K}$ if and only if the matrix M^T has the invariant cone $\mathcal{K}^*$. In this case, we have $\mathcal{K}^*(Q) = \mathcal{K}(Q^{-1})$. Thus, by virtue of the law of inertia, the obtained criterion of the type (3.6) for the invariance of the cone $\mathcal{K}(Q)$ can be represented in the form (3.4), and, furthermore, the parameter α belongs to the interval $0 \leq \alpha \leq \gamma^{-1}h^T M^T QMh$.

The theorem is proved.

Note that Theorem 3.3 generalizes the main result of [6] to ellipsoidal cones of the type $\mathcal{K}(Q)$.

Theorem 3.4. $\mathcal{K}(Q)$ *is an invariant cone of system (3.1) if and only if, for a certain* $\alpha \in R^1$, *the following matrix inequality is true:*

$$M^T Q + QM \geq \alpha Q. \tag{3.7}$$

Proof. According to (1.3), a criterion for the positivity of the system in terms of the conjugate cone $\mathcal{K}^*(Q) = \mathcal{K}(Q^{-1})$ has the form

$$z \in \mathcal{K}(Q), \quad w \in \mathcal{K}^*(Q), \quad w^T z = 0 \;\Rightarrow\; w^T Mz \geq 0. \tag{3.8}$$

We show that the orthogonality of nonzero vectors $z \in \mathcal{K}(Q)$ and $w \in \mathcal{K}^*(Q)$ implies that $w = \beta Qz$, where $\beta > 0$. Let $w = Qg$, where g is a certain vector, and let the following relations be true:

$$z^T Qz \geq 0, \quad w^T Q^{-1} w = g^T Qg \geq 0, \quad w^T z = g^T Qz = 0.$$

If $V = [z, g]$ is a matrix of full rank 2, then, for any $\varepsilon > 0$, we have

$$G_\varepsilon = V^T(Q+\varepsilon I)V = \begin{bmatrix} z^TQz & 0 \\ 0 & g^TQg \end{bmatrix} + \varepsilon V^T V > 0.$$

This implies that the vectors z and g must be linearly dependent. Otherwise, for a certain $\varepsilon > 0,$ we arrive at a contradiction:

$$1 = i_+(Q) = i_+(Q+\varepsilon I) \ge i_+(G_\varepsilon) = 2.$$

Thus, $w = \beta Qz,$ and, furthermore, $\beta > 0$ because $z^T h > 0$ and $w^T h > 0.$

Condition (3.8) means that $z^T(M^TQ + QM)z \ge 0$ for any $z \in \mathcal{K}(Q),$ which, by virtue of Lemma 3.1, is equivalent to condition (3.7).

Note that, in the case considered, we have $z \in \partial\mathcal{K}(Q),$ i.e., $z^TQz = 0.$ Therefore, condition (3.7) guarantees the invariance of the cone $\mathcal{K}(Q)$ for system (3.1) for a certain $\alpha \in R^1.$ One can show that $\alpha \le 2h^TMh.$

The theorem is proved.

Let us generalize Theorem 3.4 to the nonautonomous system

$$\dot{z} = M(t)z, \qquad t \ge 0, \tag{3.9}$$

in the phase space of which a variable ellipsoidal cone $\mathcal{K}(Q_t)$ is given. Assume that the elements of the matrices $M(t)$ and $Q_t = Q_t^T$ are continuous functions of time $t.$

Theorem 3.5. *$\mathcal{K}(Q_t)$ is an invariant cone of system (3.9) if and only if the following matrix inequality is true:*

$$\dot{Q}_t + M^T(t)Q_t + Q_tM(t) \ge \alpha(t)Q_t, \qquad t \ge 0, \tag{3.10}$$

where $\alpha(t)$ is a certain function.

Proof. Let $T(t)$ be a nonsingular matrix such that $T^T(t)Q_tT(t) \equiv \Delta.$ Using the transformation $z = T(t)w,$ we obtain the system

$$\dot{w} = N(t)w, \qquad N(t) = T^{-1}(t)M(t)T(t) - T^{-1}(t)\dot{T}(t), \tag{3.11}$$

which has an invariant circular cone $\mathcal{K}(\Delta)$ if and only if $\mathcal{K}(Q_t)$ is the invariant cone of the original system (3.9). According to Theorem 3.4, a criterion for the positivity of system (3.11) with respect to $\mathcal{K}(\Delta)$ has the form

$$N^T(t)\Delta + \Delta N(t) \ge \alpha(t)\Delta,$$

where $\alpha(t)$ is a certain function. Multiplying the last inequality from the left and from the right by $T^{-1T}(t)$ and $T^{-1}(t)$, respectively, and using the identity

$$\dot{Q}_t + T^{-1T}(t)\dot{T}^T(t)Q_t + Q_t\dot{T}(t)T^{-1}(t) \equiv 0,$$

we obtain (3.10).

The theorem is proved.

Theorem 3.6. *Suppose that there exist a symmetric matrix Q with inertia $i(Q) = \{1, n, 0\}$ and constants $\alpha \in R^1$ and $\beta > 0$ for which the following inequalities are true:*

$$\begin{gathered} M^TQ + QM \geq \alpha Q, \quad M^TQM \leq \beta Q, \\ h^TM^{-1}h \leq 0, \quad h^T(M^TQM)^{-1}h \geq 0, \end{gathered} \tag{3.12}$$

where h is the eigenvector of the matrix Q corresponding to its unique positive eigenvalue. Then the differential system (3.1) is exponentially stable and has the invariant cone $\mathcal{K}(Q)$.

This result is a corollary of Theorems 3.1, 3.3, and 3.4. An analogous statement is true for system (3.2).

Theorem 3.7. *Suppose that there exist a symmetric matrix Q with inertia $i(Q) = \{1, n, 0\}$ and constants $\alpha > 0$ and $\beta > 0$ that satisfy (3.4) and the following inequalities:*

$$M_1^TQM_1 \leq \beta Q, \quad h^TM_1^{-1}h \geq 0, \quad h^T\left(M_1^TQM_1\right)^{-1}h \geq 0, \tag{3.13}$$

where $M_1 = I - M$. Then the difference system (3.2) is asymptotically stable and has the invariant cone $\mathcal{K}(Q)$.

Example 3.1. Consider the differential system

$$\dot{z} = Mz, \quad M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5a & 4a-5 & a-4 \end{bmatrix}, \tag{3.14}$$

where a is a real parameter. Since $\sigma(M) = \{-2 \pm i, a\}$, the system has an invariant ellipsoidal cone $\mathcal{K}(Q)$ if and only if $\alpha(M) = a \in \sigma(M)$. Parallel with the matrix inequality (3.7), we consider the matrix equation

$$M^TQ + QM - \alpha Q = I. \tag{3.15}$$

By virtue of the theorem on inertia, its solution must satisfy the conditions

$$i_\alpha^+(M) = i_+(Q), \quad i_\alpha^-(M) = i_-(Q), \quad i_0(Q) = 0,$$

where $i_\alpha^+(M)$ $(i_\alpha^-(M))$ is the number of eigenvalues of the matrix M located to the right (to the left) of the straight line $2\,\mathrm{Re}\,\lambda = \alpha$. Let $\alpha = a - 2$. Then $i(Q) = \{1, 2, 0\}$.

If $a = -1$, then $\alpha = -3$, and, using Eq. (3.15), we obtain

$$Q = \begin{bmatrix} 27 & 17 & 8 \\ 17 & -3.8 & 1.2 \\ 8 & 1.2 & 0.2 \end{bmatrix}, \quad h = \begin{bmatrix} 0.89719 \\ 0.38737 \\ 0.21211 \end{bmatrix}, \quad i(Q) = \{1, 2, 0\}.$$

Solving the system of inequalities (3.12) with respect to a, α, and β for obtained Q and h, we get $a = -0.81697$, $\alpha = -2.9682$, and $\beta = 1.07585$. Thus, the conditions of Theorem 3.6 under which system (3.14) is exponentially stable and has the invariant cone $\mathcal{K}(Q)$ are satisfied.

We establish conditions for the invariance of cones of the type $\mathcal{K}_p(\mu_\alpha)$ for system (3.2) using the block representation of the matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad A = \| a_{ij} \|_1^n, \quad B = \| b_{ij} \|_1^{n,m},$$

$$C = \| c_{ij} \|_1^{m,n}, \quad D = \| d_{ij} \|_1^m.$$

Denote the kth column and the sth row of an arbitrary matrix X by x_{*k} and x_{s*}^T, respectively. Let us find conditions under which

$$\| x \|_p \le \mu_\alpha(u) \;\Rightarrow\; \| Ax + Bu \|_p \le \mu_\alpha(Cx + Du).$$

Taking into account the relation

$$\| Ax + Bu \|_p \le \| A \|_p \| x \|_p + \sum_{j=1}^m | b_{*j} | u_j \le h^T u,$$

$$\| A \|_p = \sup_{\| x \|_p = 1} \| Ax \|_p, \quad h^T = \left[\frac{\alpha}{m} \| A \|_p + \| b_{*1} \|_p, \ldots, \frac{\alpha}{m} \| A \|_p + \| b_{*m} \|_p \right], \quad \mathcal{K}_p^*(\mu_\alpha) = \mathcal{K}_q(\sigma_\beta),$$

we try to satisfy the inequalities

$$\alpha c_{s*}^T x + (\alpha d_{s*} - h)^T u \ge 0, \quad \alpha \| c_{s*} \|_q \le \sigma_\beta(\alpha d_{s*} - h), \quad s = \overline{1, m}.$$

Thus, conditions for the invariance of the cone $\mathcal{K}_p(\mu_\alpha)$ for system (3.2) have the form

$$\| A \|_p + \beta \sum_{j=1}^{m} \| b_{*j} \|_p + \alpha \| c_{s*} \|_q \leq \sum_{j=1}^{m} d_{sj},$$

$$d_{sj} \geq \frac{1}{m} \| A \|_p + \beta \| b_{*j} \|_p, \quad s = \overline{1, m}, \quad j = \overline{1, m},$$

where $p > 1$, $q > 1$, $\alpha\beta = 1$, $1/p + 1/q = 1$, and $\| A \|_p$ is the matrix norm consistent with the vector norm $\| x \|_p$. In particular,

$$\| A \|_1 = \max_j \sum_i |a_{ij}|, \quad \| A \|_2 = \left(\sum_{i,j} a_{ij}^2 \right)^{1/2}, \quad \| A \|_\infty = \max_i \sum_j |a_{ij}|.$$

In the cases $p = 2$ and $p = \infty$, we have the criteria

$$M\mathcal{K}_2(\mu_\alpha) \subset \mathcal{K}_2(\mu_\alpha) \Leftrightarrow l_j \in \mathcal{K}_2(\mu_\alpha), \quad M_k \Delta M_k^T \geq \alpha_k \Delta,$$

$$M\mathcal{K}_\infty(\mu_\alpha) \subset \mathcal{K}_\infty(\mu_\alpha) \Leftrightarrow d_{kj} \geq \beta |b_{sj}|, \quad \sum_{i=1}^{n} |\alpha c_{ki} \pm a_{si}| \leq \sum_{j=1}^{m} (d_{kj} \pm \beta b_{sj}),$$

where

$$l_j = \begin{bmatrix} b_{*j} \\ d_{*j} \end{bmatrix}, \quad M_k = \begin{bmatrix} A & \beta \sum_j b_{*j} \\ \alpha c_{k*}^T & \sum_j d_{kj} \end{bmatrix}, \quad \alpha_k \geq 0, \quad s = \overline{1, n}, \quad j, k = \overline{1, m}.$$

For $p = 2$, the following sufficient conditions can also be established:

$$Q_k > 0, \quad Q_k \geq \sum_{i=1}^{n} P_{ki} Q_k^{-1} P_{ki}, \quad k = \overline{1, m} \Rightarrow M\mathcal{K}_2(\mu_\alpha) \subset \mathcal{K}_2(\mu_\beta),$$

where

$$Q_k = \beta \sum_{j=1}^{m} Q_{kj}, \quad P_{ki} = \begin{bmatrix} \alpha c_{ki} I & a_{*i} \\ a_{*i}^T & \alpha c_{ki} \end{bmatrix}, \quad Q_{kj} = \begin{bmatrix} \alpha d_{kj} I & b_{*j} \\ b_{*j}^T & \alpha d_{kj} \end{bmatrix}.$$

In the proof, we use the representations of the cones $\mathcal{K}_2(\mu_\alpha)$ and $\mathcal{K}_2(\sigma_\beta)$ in terms of nonnegative-definite matrices, in particular

$$\mathcal{K}_2(\mu_\alpha) = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} : \begin{bmatrix} \mu_\alpha(u) I & x \\ x^T & \mu_\alpha(u) \end{bmatrix} \geq 0 \right\}.$$

Conditions for the positivity and monotonicity of nonlinear differential systems with respect to the cone $\mathcal{K}_2(\mu_\alpha)$ were given in [17].

4. Positivity and Stability of Linear Differential Equations of the sth Order

Consider a differential system of the sth order

$$A_0 x(t) + A_1 x^{(1)}(t) + \ldots + A_s x^{(s)}(t) = 0, \qquad x^{(i)}(0) = x_0^{(i)}, \quad i = \overline{0, s-1}, \tag{4.1}$$

where $x(t) \in R^n$ is the vector of phase coordinates, $t \ge 0$, and $A_i \in R^{n \times n}$ are the coefficients of a regular matrix polynomial $F(\lambda) = A_0 + \lambda A_1 + \ldots + \lambda^s A_s$. The total state of system (4.1) is characterized by the vector function $y(t)$ that is a solution of the first-order differential system

$$A\, y(t) = B\, \dot{y}(t), \qquad y(0) = y_0, \quad t \ge 0, \tag{4.2}$$

where

$$A = \begin{bmatrix} -A_0 & 0 & \ldots & 0 \\ 0 & I & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & I \end{bmatrix}, \quad B = \begin{bmatrix} A_1 & \ldots & A_{s-1} & A_s \\ I & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & I & 0 \end{bmatrix}, \quad y(t) = \begin{bmatrix} x(t) \\ x^{(1)}(t) \\ \vdots \\ x^{(s-1)}(t) \end{bmatrix}.$$

For this reason, we determine invariant sets and properties of positivity of this system with respect to cones in the phase space R^{ns}:

$$y(0) = \begin{bmatrix} x(0) \\ x^{(1)}(0) \\ \vdots \\ x^{(s-1)}(0) \end{bmatrix} \in \hat{\mathcal{K}} \;\Rightarrow\; y(t) = \begin{bmatrix} x(t) \\ x^{(1)}(t) \\ \vdots \\ x^{(s-1)}(t) \end{bmatrix} \in \hat{\mathcal{K}},$$

$$\hat{\mathcal{K}} = \begin{bmatrix} \mathcal{K}_0 \\ \mathcal{K}_1 \\ \vdots \\ \mathcal{K}_{s-1} \end{bmatrix}, \qquad t \ge 0.$$

The values of $x(t)$ belong to the set $\mathcal{K}_0$.

Let (U, T) be an arbitrary (right) eigenpair of the matrix polynomial $F(\lambda)$ determined from the conditions [4]

$$A_0 T + A_1 TU + \ldots + A_s TU^s = 0, \qquad \operatorname{rank} E = m, \qquad E \overset{\Delta}{=} \begin{bmatrix} T \\ TU \\ \vdots \\ TU^{s-1} \end{bmatrix}, \tag{4.3}$$

where $T \in C^{n\times m}$ and $U \in C^{m\times m}$. Then the spectrum of the matrix U is a subset of the spectrum $\sigma(F)$ of the matrix polynomial $F(\lambda)$. It is also known that (U, T) is an eigenpair of $F(\lambda)$ if and only if (U, E) is an eigenpair of the linear pencil $L(\lambda) = A - \lambda B$, i.e.,

$$AE = BEU, \qquad \operatorname{rank} E = m. \tag{4.4}$$

Note that the spectra $\sigma(L)$ and $\sigma(F)$ coincide.

An eigenpair (U, T) of the matrix polynomial $F(\lambda)$ is called maximal if the number m in (4.3) takes its maximum possible value. If (U, T) is a maximal eigenpair of the matrix polynomial $F(\lambda)$, then m coincides with the number of eigenvalues of $F(\lambda)$ (counting multiplicity).

Lemma 4.1. *An eigenpair (U, T) of the matrix polynomial $F(\lambda)$ is maximal if and only if the following conditions are satisfied:*

$$\operatorname{rank}[\, F(\lambda), \Phi(\lambda)\,] \equiv n, \qquad \Phi(\lambda) \overset{\Delta}{=} \sum_{i=1}^{s} \lambda^{i-1} \sum_{j=i}^{s} A_j TU^{j-i}, \quad \lambda \in \sigma(F). \tag{4.5}$$

Proof. Let (U, T) be a maximal eigenpair of the matrix polynomial $F(\lambda)$. Then (U, E) is a maximal eigenpair of the pencil $L(\lambda)$. We now use the Kronecker canonical form of a regular pencil [18] and the structure of the matrix E in (4.4), namely,

$$P(A - \lambda B)Q = \begin{bmatrix} J - \lambda I & 0 \\ 0 & I - \lambda N \end{bmatrix}, \qquad E = Q\begin{bmatrix} R \\ 0 \end{bmatrix}, \qquad JR = RU, \tag{4.6}$$

where $J \in C^{m\times m}$, $\sigma(J) = \sigma(L)$, P and Q are nonsingular matrices, and N is a nilpotent matrix all elements of which are equal to zero, except possibly for unities on the principal superdiagonal. In the case considered, R is a nonsingular square matrix and one can easily establish the identity

$$\operatorname{rank}\,[A - \lambda B, BE] \equiv ns, \qquad \lambda \in C^1,$$

which can be reduced to the form (4.5) by equivalent block transformations.

Conditions (4.5) can be rewritten in the form

$$v^T F(\lambda) = 0, \quad v \neq 0 \;\Rightarrow\; v^T \Phi(\lambda) \neq 0, \quad \lambda \in F(\lambda),$$

and the matrix equation in (4.3) is equivalent to the identity

$$F(\lambda)T \equiv \Phi(\lambda)(\lambda I - U), \quad \lambda \in C^1.$$

Let v^T be the left eigenvector of the matrix polynomial $F(\lambda)$ that corresponds to an eigenvalue $\lambda \in \sigma(F)$. Under conditions (4.5), it follows from the last identity that $u^T = v^T\Phi(\lambda)$ is the left eigenvector of the matrix U that corresponds to its eigenvalue $\lambda \in \sigma(U)$. This means that (U, T) is a maximal eigenpair of the matrix polynomial $F(\lambda)$.

The lemma is proved.

The statement presented below can be useful for the numerical determination of eigenpairs of a matrix polynomial.

Lemma 4.2. *If $n \times m$ matrices $R_0, \dots, R_s$ satisfy the conditions*

$$A_0R_0 + A_1R_1 + \dots + A_sR_s = 0, \tag{4.7}$$

$$\operatorname{rank} S_0 = \operatorname{rank}[S_0, S_1] = m \le sn, \qquad S_0 \stackrel{\Delta}{=} \begin{bmatrix} R_0 \\ \vdots \\ R_{s-1} \end{bmatrix}, \qquad S_1 \stackrel{\Delta}{=} \begin{bmatrix} R_1 \\ \vdots \\ R_s \end{bmatrix}, \tag{4.8}$$

then the matrices

$$U = \left(S_0^T S_0\right)^{-1} S_0^T S_1, \qquad T = R_0, \tag{4.9}$$

form an eigenpair of the matrix polynomial $F(\lambda)$, i.e., they satisfy relations (4.3).

Proof. Under conditions (4.8), there exists a unique solution of the equation $S_0 U = S_1$ determined by (4.9). Moreover, S_0 coincides with E, and the matrix equation (4.7) reduces to the form (4.3).

The lemma is proved.

Lemma 4.3. *Let (U, T) be an eigenpair of the matrix polynomial $F(\lambda)$. In this case, $\hat{\mathcal{K}} = E\mathcal{K}$ is an invariant set of system (4.1) if and only if $\mathcal{K}$ is an invariant set of the system*

$$\dot{z} = Uz, \qquad z(0) = z_0, \qquad t \ge 0. \tag{4.10}$$

In particular, system (4.1) is positive with respect to the cone $\hat{\mathcal{K}} = E\mathcal{K}$ only if system (4.10) is positive with respect to the cone $\mathcal{K}$.

Proof. We construct a solution of system (4.2) in the form $y(t) = Ez(t)$. Taking (4.4) and (4.6) into account, we obtain

$$BE(\dot{z} - Uz) = 0, \qquad BE = P^{-1}\begin{bmatrix} R \\ 0 \end{bmatrix}.$$

Since $\operatorname{rank}(BE) = \operatorname{rank} E = m$, we conclude that $y(t)$ is a solution of system (4.2) if and only if $z(t)$ satisfies (4.10). Therefore, system (4.2) [and, hence, system (4.1)] has an invariant cone of the type $E\mathcal{K}$ only if $\mathcal{K}$ is an invariant cone of system (4.10).

The lemma is proved.

The statement below is a corollary of Theorem 3.11 and the fact that a maximal pair of a matrix polynomial completely determines its spectrum, i.e., $\sigma(U) = \sigma(F)$.

Theorem 4.1. *Let* (U, T) *be a maximal eigenpair of the matrix polynomial* $F(\lambda)$ *such that system (4.10) is positive with respect to the solid cone* $\mathcal{K}$. *Then the following assertions are equivalent:*

(1) *system (4.1) is exponentially stable;*

(2) $\operatorname{Re}\lambda < 0 \quad \forall\, \lambda \in \sigma(U)$;

(3) $\mathcal{K} \subset -U\mathcal{K}$;

(4) $\exists\, z_0 \in \operatorname{int}\mathcal{K} \colon \; Uz_0 \in -\operatorname{int}\mathcal{K}$.

We now formulate sufficient conditions for the positivity and exponential stability of system (4.1). Assume that

$$B\hat{\mathcal{K}} \subset (\gamma B - A)\hat{\mathcal{K}} \quad \forall\, \gamma \geq 0, \tag{4.11}$$

where $\hat{\mathcal{K}} \subset R^{ns}$ is a certain set. Then, according to (4.6), we have

$$\hat{\mathcal{K}}_1 \subset (\gamma I - J)\hat{\mathcal{K}}_1, \qquad -(N + \gamma N^2 + \ldots + \gamma^{\nu-2} N^{\nu-1})\hat{\mathcal{K}}_2 \subset \hat{\mathcal{K}}_2,$$

where $\hat{\mathcal{K}} = Q_1\hat{\mathcal{K}}_1 + Q_2\hat{\mathcal{K}}_2$, $Q = [Q_1, Q_2]$, and ν is the nilpotency index of the matrix N. The set $\hat{\mathcal{K}}$ is invariant for system (4.2) only if $\hat{\mathcal{K}}_2 = \{0\}$. If $\hat{\mathcal{K}}_1$ is a normal reproducing cone, then, according to Theorem 3.1, the first inclusion guarantees the positivity of the system $\dot{z} = Jz$ with respect to $\hat{\mathcal{K}}_1$ and its exponential stability. In this case, system (4.2) is exponentially stable and has the invariant set $\hat{\mathcal{K}} = Q_1\hat{\mathcal{K}}_1$, which is a cone of dimension $\dim\hat{\mathcal{K}} = m$ only if $\hat{\mathcal{K}}_1$ is a reproducing cone. The set $\hat{\mathcal{K}} = M_\alpha^\nu\mathcal{K} = Q_1(\alpha I - J)^{-\nu}\mathcal{K}_1$, where $M_\alpha \overset{\Delta}{=} (\alpha B - A)^{-1}B$ and $\mathcal{K} = Q_1\mathcal{K}_1 + Q_2\mathcal{K}_2$, $\alpha \notin \sigma(F)$, has an analogous structure. In particular, we can set $\alpha = 0$.

Using Theorem 3.2 and the arguments presented above, we obtain the following statements:

Theorem 4.2. *Suppose that, for a certain* γ_0, *the following conditions are satisfied:*

$$B\hat{\mathcal{K}} \subset -A\hat{\mathcal{K}} \cap (\gamma_0 B - A)\hat{\mathcal{K}}, \qquad \gamma_0 > \frac{\rho^2(F) - r^2(F)}{2r(F)}, \tag{4.12}$$

where $\rho(F) \overset{\Delta}{=} \max\{|\lambda| : \lambda \in \sigma(F)\}$, $r(F) \overset{\Delta}{=} \min\{|\lambda| : \lambda \in \sigma(F)\}$, *and* $\hat{\mathcal{K}} = M_\alpha^\vee \mathcal{K}$ *is a normal cone of dimension* m. *Then system (4.1) is exponentially stable.*

Theorem 4.3. *If, for a certain maximal eigenpair* (U, T) *of the matrix polynomial* $F(\lambda)$, *one has*

$$\mathcal{K} \subset (\gamma I - U)\mathcal{K} \quad \forall\, \gamma \geq 0, \tag{4.13}$$

where $\mathcal{K}$ *is a normal reproducing cone, then system (4.1) is exponentially stable and has the invariant cone* $\hat{\mathcal{K}} = E\mathcal{K}$.

Note that inclusion (4.13) follows from (4.11) if we set $\hat{\mathcal{K}} = E\mathcal{K} = Q_1 R\mathcal{K}$ and take the equality $JR = RU$ into account.

Example 4.1. Consider the second-order differential system

$$A_0 x + A_1 \dot{x} + A_2 \ddot{x} = 0, \tag{4.14}$$

where

$$A_0 = \begin{bmatrix} 8 & 1 \\ -9 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 4 & 1 \\ -4 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

This system is associated with the matrix quadratic pencil

$$F(\lambda) = A_0 + \lambda A_1 + \lambda^2 A_2 = \begin{bmatrix} \lambda^2 + 4\lambda + 8 & \lambda + 1 \\ -4\lambda - 9 & \lambda + 1 \end{bmatrix}$$

with spectrum $\sigma(F) = \{-4 \pm i, -1\}$. Using the “Given … Find” construction in the MATHCAD package, we determine the following maximal eigenpair of this pencil:

$$U = \begin{bmatrix} -1.525 & 0.53 & 0 \\ 0.688 & -3.686 & 1.595 \\ 3.448 & 0 & -3.79 \end{bmatrix}, \quad T = \begin{bmatrix} -0.13 & 0.26 & -0.103 \\ 3.299 & 1.309 & -0.073 \end{bmatrix}.$$

The off-diagonal elements of the matrix U are nonnegative, and, for its inverse, we have

$$U^{-1} = \begin{bmatrix} -0.822 & -0.118 & -0.05 \\ -0.476 & -0.34 & -0.143 \\ -0.748 & -0.108 & -0.309 \end{bmatrix} \overset{\mathcal{K}}{\leq} 0,$$

where $\mathcal{K} = R_+^3$ is the cone of nonnegative vectors.

Thus, the conditions of Theorem 4.1 are satisfied and system (4.14) is exponentially stable. Moreover, by virtue of Lemma 4.3, it has the invariant cone

$$\hat{\mathcal{K}} = \begin{bmatrix} \mathcal{K}_0 \\ \mathcal{K}_1 \end{bmatrix}, \quad \mathcal{K}_0 = T\mathcal{K}, \quad \mathcal{K}_1 = TU\mathcal{K}. \tag{4.15}$$

Using Theorem 3.4, we determine the maximal eigenpair (U, T) of the quadratic pencil $F(\lambda)$ and the parameters of the ellipsoidal cone $\mathcal{K}(Q) \subset R^3$ that satisfy Theorem 4.1. The system of inequalities

$$U^T Q + QU \geq \alpha Q, \quad U^T QU \leq \beta Q, \quad h^T U^{-1} h \leq 0, \quad h^T (U^T QU)^{-1} h \geq 0,$$

is satisfied for the following values of parameters:

$$U = \begin{bmatrix} -1.67 & 0.479 & 0.037 \\ 0.071 & -3.343 & 1.028 \\ 7.627 & 0.245 & -3.987 \end{bmatrix}, \quad T = \begin{bmatrix} 0.148 & 0.022 & -0.066 \\ 2.208 & 0.395 & -0.027 \end{bmatrix},$$

$$Q = \begin{bmatrix} 27 & 17 & 8 \\ 17 & -3.8 & 1.2 \\ 8 & 1.2 & 0.2 \end{bmatrix}, \quad h = \begin{bmatrix} 0.897 \\ 0.387 \\ 0.212 \end{bmatrix}, \quad \alpha = -2.96, \quad \beta = 1.115.$$

Here, (U, T) is a maximal eigenpair of $F(\lambda)$, $i(Q) = \{1, 2, 0\}$, and h is the eigenvector of the matrix Q that corresponds to its unique positive eigenvalue.

Thus, according to Theorem 4.1, system (4.14) is exponentially stable and has an invariant cone of the type (4.15), where $\mathcal{K} = \mathcal{K}(Q)$ is an ellipsoidal cone.

This work was partially supported by NDR (grant. No. 0105U00108).

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